THE GEOMETRIC SUBGROUP MEMBERSHIP PROBLEM

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Abstract. We show that every infinite graph which is locally finite and connected admits a translation-like action by $\mathbb{Z}$ such that the distance between a vertex v and $v * 1$ is uniformly bounded by 3. This action can be taken to be transitive if and only if the graph has one or two ends. This strengthens a theorem by Brandon Seward.

Our proof is constructive, and thus it can be made computable. More precisely, we show that a finitely generated group with decidable word problem admits a translation-like action by $\mathbb{Z}$ which is computable, and satisfies an extra condition which we call decidable orbit membership problem.

As an application we show that on any finitely generated infinite group with decidable word problem, effective subshifts attain all effectively closed Medvedev degrees. This extends a classification proved by Joseph Miller for $\mathbb{Z}^d$, $d \geq 1$.

1. Introduction

1.1. **Translation-like actions by $\mathbb{Z}$ on locally finite graphs.** A right action $*$ of a group H on a metric space (X, d) is called a **translation-like action** if it is **free**[1], and for each $h \in H$, the set $\{d(x, x * h) | \ x \in X\}$ is bounded. If G is a finitely generated group endowed with the word metric associated to some finite set of generators, then the action of any subgroup H on G by right translations $(g, h) \mapsto gh$ is a translation-like action. Thus, translation-like actions can be regarded a generalization of subgroup containment.

Following this idea, Kevin Whyte proposed in [29] to replace subgroups by translation-like actions in different questions or conjectures about groups and subgroups, and called these geometric reformulations. For example, the von Neumann Conjecture asserted that a group is nonamenable if and only if it contains a nonabelian free subgroup. Its geometric reformulation asserts then that a group is nonamenable if and only if it admits a translation-like action by a nonabelian free group. While the conjecture was proven to be false [23], Kevin Whyte proved that its geometric reformulation holds.

One problem left open in [29] was the geometric reformulation of Burnside's problem. This problem asked if every finitely generated infinite group contains $\mathbb{Z}$ as a subgroup, and was answered negatively in [12]. Brandon Seward proved that the geometric reformulation of this problem also holds.

Theorem 1.1 (Geometric Burnside's problem, [27]). *Every finitely generated infinite group admits a translation-like action by $\mathbb{Z}$.*

A finitely generated infinite group with two or more ends has a subgroup isomorphic to $\mathbb{Z}$ by Stalling's structure theorem. Thus, it is the one ended case that makes necessary the use of translation-like actions. In order to prove Theorem 1.1, Brandon Seward proved a more general graph theoretic result.

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[1] Which means that $x * h = x$ implies $h = 1_H$, for $x \in X$, $h \in H$.

Theorem 1.2 ([27, Theorem 1.6]). *Let Γ be an infinite graph whose vertices have uniformly bounded degree. Then Γ admits a transitive translation-like action by $\mathbb{Z}$ if and only if it is connected and has one or two ends.*

This result proves Theorem 1.1 for one ended groups, and indeed it says more, as the translation-like action obtained is transitive. The proof of this result relies strongly on the hypothesis of having uniformly bounded degree. Indeed, the uniform bound on $d_\Gamma(v, v * 1)$ depends linearly on a uniform bound for the degree of the vertices of the graph.

We will strengthen this result by weakening the hypothesis to the locally finite case, and improving the uniform bound on $d_\Gamma(v, v * 1)$ to 3.

Theorem 1.3. *Let Γ be an infinite graph whose vertices have finite degree. Then Γ admits a transitive translation-like action by $\mathbb{Z}$ if and only if it is connected and has one or two ends.*

Moreover, the action can be taken such that the distance between a vertex v and $v * 1$ is uniformly bounded by 3.

A problem left in [27, Problem 3.5] was to characterize which graphs admit a transitive translation-like action by $\mathbb{Z}$. Thus we have solved the case of locally finite graphs, and it only remains the case of graphs with vertices of infinite degree.

We now mention an application of translation-like actions to Cayley graphs and Hamiltonian paths. This is related to a special case of Lovász conjecture which asserts the following. If G is a finite group, then for every set of generators the associated Cayley graph admits a Hamiltonian path. Note that the existence of at least one such generating set is obvious ($S = G$), and the difficulty of the question, which is still open, is that it alludes every generating set. Now assume that G is an infinite group, S is a finite set of generators, and $\mathrm{Cay}(G, S)$ admits a transitive translation-like action by $\mathbb{Z}$. This action becomes a bi-infinite Hamiltonian path after we enlarge the generating set, and thus it follows from Seward's theorem that every finitely generated group with one or two ends admits a generating set for which the associated Cayley graph admits a bi-infinite Hamiltonian path [27, Theorem 1.8]. It is not known whether this holds for every Cayley graph [27, Problem 4.8], but our result yields an improvement in this direction.

Corollary 1.4. *Let G be a finitely generated group with one or two ends, and let S be a finite set of generators. Then the Cayley graph of G with respect to the generating set $\{g \in G | \ d_S(g, 1_G) \leq 3\}$ admits a bi-infinite Hamiltonian path.*

This was known to hold for generating sets of the form $\{g \in G | \ d_S(g, 1_G) \leq J\}$, where $S \subset G$ is a finite generating set for G and J depends linearly on the vertex degree of $\mathrm{Cay}(G, S)$.

In the more general case where we impose no restrictions on ends, we obtain the following result for non transitive translation-like actions. This readiliy implies Theorem 1.1.

Theorem 1.5. *Every infinite graph which is locally finite and connected admits a translation-like action by $\mathbb{Z}$ such that the distance between a vertex v and $v * 1$ is uniformly bounded by 3.*

These statements about translation-like actions can also be stated in terms of powers of graphs. Given a graph Γ, its n-th power Γ^n is defined as the graph with the same set of vertices, and where two vertices u, v are joined if their distance in Γ is at most n. It was shown by Jerome Karaganis [19] that the cube of a finite and connected graph is Hamiltonian. Our Theorem 1.3 generalizes this to infinite graphs, that is, the cube of a locally finite connected graph with one or two ends admits a bi-infinite Hamiltonian path. In our proofs we make use of Karaganis result: we will define bi-infinite Hamiltonian paths locally.

We mention that Theorem 1.5 has been proved in [6, Section 4], using the same fact about cubes of finite graphs.

1.2. **Computability of translation-like actions.** Let us now turn our attention to the problem of the computability of translation-like actions on groups or graphs. In order to discuss this precisely, we need the following notions. A graph Γ is computable if there exists an algorithm which given two vertices u and v, determines whether they are neighbors or not. A stronger notion is that of a highly computable graph, namely, a computable graph which is locally finite and for which the function which maps a vertex to its degree is computable. This extra condition is necessary to compute the neighborhood of a vertex.

An important example comes from group theory: if G is a finitely generated group with decidable word problem and S is a finite set of generators, then its Cayley graph with respect to S is highly computable.

A classical example of a problem in graph theory which admits no computable solution is that of finding infinite paths. Kőnig's infinity lemma asserts that every infinite, connected, and locally finite graph admits an infinite path. However, there are highly computable graphs which admit paths, all of which are uncomputable. Another example comes from Hall's matching theorem. There are highly computable graphs satisfying the hypothesis in the theorem, but which admit no computable right perfect matching [21]. These two results are used in the proof of Seward's theorem, so the translation-like actions from this proof are not clearly computable.

By a computable translation-like action on a computable graph Γ, we mean that the function $* : V(\Gamma) \times \mathbb{Z} \to V(\Gamma)$ is computable, i.e. there exists an algorithm which given a vertex $v \in V(\Gamma)$ and $n \in \mathbb{Z}$, computes the vertex $v * n$.

In Section 4 we will provide a computable version of Theorem 1.3, from which follows that a highly computable graph admits a transitive translation-like action by $\mathbb{Z}$ if and only if it admits a computable one. The restriction on ends is essential to make the proof computable, and indeed we can not show a computable version of Theorem 1.5.

Our interest in the computability of translation-like actions comes from symbolic dynamics, and the shift spaces associated to a group. We will need a computable translation-like action such that it is possible to distinguish in a computable manner between different orbits. Let us introduce a general definition, though we will only treat the case where the acting group is $\mathbb{Z}$.

Definition 1.6. Let G be a group, and $S \subset G$ a finite set of generators. A group action of H on G is said to have **decidable orbit membership problem** if there exists an algorithm which given two words u and v in $(S \cup S^{-1})^*$, decides whether the corresponding group elements u_G, v_G lie in the same orbit under the action.

This property arises when one requires a standard construction in the shift space A^G to preserve a complexity measure called Medvedev degree. This is the motivation behind the following result, whose application is discussed below (Theorem 1.8).

Theorem 1.7. *Let G be a finitely generated infinite group with decidable word problem. Then G admits a computable translation-like action by $\mathbb{Z}$ with decidable orbit membership problem.*

It is interesting to note that the introduced property corresponds to the geometric reformulation of a subgroup property: the decidable subgroup membership problem. If G is a finitely generated group with decidable word problem, then the action of a subgroup H by right translations has decidable orbit membership problem if and only if H has decidable subgroup membership problem (Proposition 4.10).

The proof of Theorem 1.7 proceeds as follows. For groups with one or two ends, we will show the existence of a computable and transitive translation-like action by $\mathbb{Z}$, i.e. a computable version of Theorem 1.3. This has decidable orbit membership problem for the trivial reason that has only one orbit. For groups with two or more ends, we will obtain the desired translation-like action from a subgroup isomorphic to $\mathbb{Z}$ with decidable membership problem. The existence of this subgroup is a consequence of the computability of the normal form from Stalling's structure theorem on groups with decidable word problem (Proposition 4.8).

1.3. **Medvedev degrees of effective subshifts.** Let us now turn our attention to Medvedev degrees, a complexity measure which is defined using computable functions. Precise definitions of this and the following concepts are given in Section 5. Intuitively, the Medvedev degree of a set $P \subset A^{\mathbb{N}}$ measures how hard is to compute a point in P. For example, a set has zero Medvedev degree if and only if it has a computable point. Note that this becomes meaningful if we regard P as the set of solutions to some problem.

This notion can be applied to a variety of objects, such as graph colorings [25], paths on graphs, matchings from Hall's matching theorem, and others [10, Chapter 13]. In this article we consider Medvedev degrees of subshifts.

Let G be a finitely generated group, and let A be a finite alphabet. A subshift is a subset of A^G which is closed in the prodiscrete topology, and is invariant under translations. Dynamical properties of subshifts have been related to their computational properties in different ways. A remarkable example of this is the characterization of the entropies of two dimensional subshifts of finite type as the class of nonnegative Π_1^0 real numbers [15].

In the same manner that it is done with the entropy, one may ask which is the class of Medvedev degrees that a certain class of subshifts can attain. The classification is known for subshifts of finite type in $\mathbb{Z}^d, d \geq 1$. In the case $d = 1$, all subshifts of finite type have Medvedev degree zero, because all of them contain a periodic point, and then a computable point. In the case $d \geq 2$, subshifts of finite type can attain the class of Π_1^0 Medvedev degrees [28].

A larger class of subshifts is that of effective subshifts. A subshift $X \subset A^{\mathbb{Z}}$ is effective if the set of words which do not appear in its configurations is computably enumerable. This notion can be extended to a finitely generated group, despite some intricacies that arise in relation to the word problem of the group. In this article we deal with groups with decidable word problem, and the notion of effective subshift is a straightforward generalization.

Answering a question left open in [28], Joseph Miller proved that effective subshifts over $\mathbb{Z}$ can attain all Π_1^0 Medvedev degrees [22]. We generalize this result to the class of infinite, finitely generated groups with decidable word problem.

Theorem 1.8. *Let G be a finitely generated and infinite group with decidable word problem. The class of Medvedev degrees of effective subshifts on G is the class of all Π_1^0 Medvedev degrees.*

The idea for the proof is the following. Given any subshift $Y \subset A^{\mathbb{Z}}$, we can produce a new subshift $X \subset B^G$ that simultaneously describes translation-like actions by $\mathbb{Z}$, and elements in Y. Then Theorem 1.7 ensures that this construction preserves the Medvedev degree of Y, and the result follows from the known classification for $\mathbb{Z}$.

Despite the simplicity of the proof we need to translate some computability notions from $A^{\mathbb{N}}$ to A^G, this is done by taking a computable numbering of G. The notions obtained do not depend of the numbering, and are consistent with notions present in the literature defined by other means.

The mentioned construction using translation-like actions has been used in different results in the context of symbolic dynamics. For example, to transfer results on the emptiness problem for subshifts of finite type from one group to another [18], to produce aperiodic subshifts of finite type on new groups [6, 18], and to study the entropy of subshifts of finite type on some amenable groups [3].

Paper structure. In Section 2 we fix some notation, and recall some basic facts on graph and group theory. We also fix our computability setting for countable sets. In Section 3 we show Theorem 1.3 and Theorem 1.5. In Section 4 we show Theorem 1.7, and apply this in Section 5 to prove Theorem 1.8.

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2. Preliminaries

2.1. **Graph theory.**

In this article all graphs are assumed to be undirected, without self loops, and simple (not multigraphs). We now fix some terminology and recall some basic facts.

Given a graph Γ we denote its vertex set by $V(\Gamma)$, and its edge set by $E(\Gamma)$. Each edge e is an unordered pair of vertices $\{x, y\}$, we say that e **joins** x and y, its **endpoints**. We also say that x, y are **adjacent** or **neighbors**. The **degree** of a vertex x is the number of its neighbors, and is denoted $\deg_\Gamma(x)$.

A **subgraph** of Γ is a graph whose edge and vertex set are contained in the edge and vertex set of Γ. A set of vertices $V \subset V(\Gamma)$ determines a subgraph of Γ, denoted $\Gamma[V]$, as follows. The vertex set of $\Gamma[V]$ is V, and its edge set is the set of all edges in $E(\Gamma)$ whose endpoints are both in V. We say that $\Gamma[V]$ is the subgraph of Γ **induced** by V.

Another way to obtain a subgraph of Γ is by deleting a set of vertices. For $V \subset V(\Gamma)$, $\Gamma - V$ denotes the graph $\Gamma[V(\Gamma) - V]$, that is, we erase from Γ all vertices in V, and all edges incident to them. If Γ' is a subgraph of Γ, then $\Gamma - \Gamma'$ stands for $\Gamma - V(\Gamma')$.

A **path** on Γ is an injective function $f : [n, m] \subset \mathbb{Z} \to V(\Gamma)$ which sends consecutive integers to adjacent vertices. We say that f **joins** $f(n)$ to $f(m)$, and define its **length** to be $n - m$.

We say that Γ is **connected** when any pair of vertices are joined by a path. A **connected component** of Γ is a connected subgraph of Γ which is maximal for the subgraph relation.

If Γ is connected, we define the path length metric d_Γ on $V(\Gamma)$ as follows. Given two vertices u and v, $d_\Gamma(u, v)$ is the length of the shortest path which joins u to v.

The **number of ends** of Γ is the supremum of the number of infinite connected components of $\Gamma - V$, where V ranges over all finite sets of vertices in $V(\Gamma)$. Thus the number of ends of Γ is an element of $\mathbb{N} \cup \{\infty\}$.

2.2. **Elementary computability notions.**

The following definitions of computable functions and sets are fairly standard, and can be found, for example, in [5].

Given any set A, we denote by A^* the set of all finite words of elements of A. In this context we call A an **alphabet**. The empty word is denoted by ε. A word u of length n is a **prefix** of v if they coincide in the first n symbols.

Now let A, B be finite alphabets. Recall that a **partial function** from A^* to B^* is a function whose domain is a subset of A^*, and that it is called **total** if its domain is all of A^*.

A partial function $f\colon \mathrm{Dom}(f) \subset A^* \to B^*$ is **computable** when there exists an algorithm which on input a word $w \in A^*$ in the domain of f halts and outputs $f(w)$, and which does not halt when given as input a word outside the domain of f.

A subset $X \subset A^*$ is **computable** or **decidable** if there is an algorithm that decides which elements belong to X and which elements do not, that is, its characteristic function is a total computable function.

A subset $X \subset A^*$ is **computably enumerable** or **semi decidable** if it is the domain of a partial computable function. Equivalently, there is an algorithm that halts with input x if and only if $x \in X$. Equivalently, X is the range of a partial computable function.

We say that a function or a set is **uncomputable** when it is not computable. It is easily seen that a set X is computable if and only if X and its complement are computably enumerable.

We can obtain computability notions on $\mathbb{N}^p, p \geq 1$ by identifying $\mathbb{N}^p$ with a computable subset of A^*, for some finite alphabet A. Indeed, we commonly write elements of $\mathbb{N}^p$ as words on the finite alphabet of digits from 0 to 9, the parenthesis symbols, and the comma symbol.

2.3. **Words and finitely generated groups.** It will be convenient for us to keep the distinction between words and group elements. Let G be a group, and S a subset of G. We denote by S^{-1} the set $\{s^{-1} | \ s \in S\}$. Now given a word $w \in (S \cup S^{-1})^*$ which is the concatenation of $s_1, \dots, s_n \in S$, we denote by w^{-1} the word obtained by concatenating $s_n^{-1} \dots s_1^{-1}$.

Given a word $w \in (S \cup S^{-1})^*$, we denote by w_G the group element that it represents, that is, the group element obtained replacing concatenation by the group operation. We also write $u =_G v$ if the words u, v correspond to the same group element.

A set $S \subset G$ is said to generate G if every group element can be written as a word in $(S \cup S^{-1})^*$, and G is **finitely generated** if it admits a finite generating set.

Let us assume now that S is a finite set of generators for G. The (undirected, and right) **Cayley graph** of G relative to S is denoted by $\mathrm{Cay}(G, S)$. Its vertex set is G, and two vertices g, h are joined by an edge when $gs = h$ for some $s \in (S \cup S^{-1})$. The path length metric that this graph produces on G is the same as the **word metric** associated to S**,** denoted d_S. Given a pair of elements $g, h \in G$, their distance $d_S(g, h)$ is the length of the shortest word $w \in (S \cup S^{-1})^*$ such that $gw_G = h$. We denote by $B(g, r)$ the ball $\{h \in G \mid d_S(g, h) \leq r\}$.

The number of ends of a finitely generated group is defined as the number of ends of its Cayley graph, for some generating set. This definition does not depend on the chosen generating set, and can only be among the numbers $\{0, 1, 2, \infty\}$, as proved in [11, 17].

Now let H be a subgroup of G. We say that H has **decidable subgroup membership problem** if the set $\{w \in (S \cup S^{-1})^* \mid w_G \in H\}$ is decidable. This property depends only on G and H, and not on S. In other words, this set is decidable for some fixed and finite generating set S if and only if this holds for every generating set.

In the particular case where $H = \{1_G\}$, the set defined above is called the **word problem** of G, and denoted by $\mathrm{WP}(G, S)$. This notion is related to that of computable group, which we discuss in the following subsection.

2.4. **Computability on countable sets via numberings.** In this article we need to discuss the computability of graphs, group actions, and infinite paths. For a group G, this will be related with computability notions on the symbolic space A^G in Section 5. We consider a unified approach using numberings, a generalization of Gödel numberings introduced by Yuri Ershov. A survey on the subject can be found in [13, Chapter 14].

Definition 2.1. A (bijective) **numbering** of a set X is a bijective map $\nu : N \to X$, where N is a decidable subset of $\mathbb{N}$. We call (X, ν) a **numbered set**. When $\nu(n) = x$, we say that n is a **name** for x, or that n represents x.

As long as the set X is infinite, we can always assume that the set N equals $\mathbb{N}$, because a decidable and infinite subset of $\mathbb{N}$ admits a computable bijection onto $\mathbb{N}$.

Computability notions are transferred from $\mathbb{N}$ to numbered sets as follows. A subset Y of (X, ν) is decidable or computable when $\nu^{-1}(Y) \subset \mathbb{N}$ is a decidable set. Given two numbered sets $\nu : N \to X$, $\nu : N' \to X'$, we can endow their product $X \times X'$ with a numbering as follows. The product of ν and ν' is a bijection $N \times N' \subset \mathbb{N}^2 \to X \times X'$, where $N \times N'$ is a decidable subset of $\mathbb{N}^2$. It remains to compose with a computable bijection between $N \times N'$ and a decidable subset $N'' \subset \mathbb{N}$, and we obtain a numbering of $X \times X'$.

This gives a definition of a decidable or computable relation $R \subset X \times X'$ between numbered sets (X, ν) and (X', ν'). In the particular case where the relation is a function $f : X \to X'$, this is equivalent to ask $\nu'^{-1} \circ f \circ \nu : \mathbb{N} \to \mathbb{N}$ to be computable.

Different numberings on a set X may produce different computability notions[2]. However after we impose some structure on the set -such as functions or relations- and require the numbering to make them computable, the possibilities can be dramatically reduced. A notable example is that of finitely generated groups, as we will see later. The following two definitions will be relevant to us.

Definition 2.2. A computable group is a group $(G, \star)$ and a numbering ν of G such that the function $\star : G^2 \to G$ is computable. A numbering of a group is **computable** if it satisfies the previous condition.

For example, $\mathbb{Z}$ admits a computable numbering, and thus is a computable group.

Definition 2.3. A **computable graph** is an undirected graph Γ and a numbering of $V(\Gamma)$ such that the adjacency relation is decidable, i.e. $\{(x, y) \mid \{x, y\} \in E(\Gamma)\}$ is a decidable subset of $V(\Gamma)^2$. A computable graph is **highly computable** if it is locally finite and the function $V \to \mathbb{N}$, $v \mapsto \deg(v)$ is computable.

Now a computable graph or group defines a class of computable objects. For example, a bi-infinite path $f : \mathbb{Z} \to V(\Gamma)$ is computable if the function f is computable between numbered sets $\mathbb{Z}$ and $V(\Gamma)$, and so on.

Let us now discuss some properties of computable groups and their numberings. The following proposition will be used to define numberings easily.

Proposition 2.4. *Let $R \subset \mathbb{N}^2$ be a decidable equivalence relation. Then the set of equivalence classes $\mathbb{N}^2/R$ admits a unique numbering, up to equivalence, such that the relation $\in$ is decidable.*

[2] Indeed, non equivalent numberings do admit a semilattice structure. See [2] and references therein.

To prove the existence we can just represent each equivalence class by the least natural number in that equivalence class, the full proof is given at the end of this subsection.

Thus to define a numbering on any set X, it is enough to take a surjection $s : \mathbb{N} \to X$ such that we can decide if two numbers n, m name the same element (so the relation $s(n) = s(m)$ is decidable), and this numbering enjoys a uniqueness property. This can be applied in a variety of situations.

Now let G be a group with decidable word problem, $S \subset G$ a finite set of group generators, and $\pi : (S \cup S^{-1})^* \to G$ the map $w \mapsto w_G$. The equivalence relation $\pi(u) = \pi(v)$ is decidable, as two words u and v have the same image under π if and only if uv^{-1} lie in the decidable set $\mathrm{WP}(G, S)$. We obtain a numbering of G by Proposition 2.4. Indeed, this is a computable numbering, and it is the only one up to equivalence:

Proposition 2.5 ([24]). *A finitely generated group admits a computable numbering if and only if it has decidable word problem. If this is the case, then all its computable numberings are equivalent.*

Moreover, computability notions are preserved by group isomorphisms.

Proposition 2.6. *Let G and G' be finitely generated groups with decidable word problem, and $f : G \to G'$ a group isomorphism. Then f is computable (for any computable numbering of G and G').*

The proofs are also given below. Let us now review some properties of highly computable graphs, let Γ be a highly computable graph. The crucial consequence of this hypothesis is that given some vertex v, and $r \in \mathbb{N}$, we can compute (uniformly on v and r) all vertices in the ball $\{v \in V(\Gamma) \mid d_\Gamma(v, v_0) \leq r\}$. This is true for $r = 1$: as the adjacency relation is decidable, we can computably enumerate the set $\{v \in V(\Gamma) \mid d_\Gamma(v, v_0) \leq 1\}$, and $\deg_\Gamma(v_0)$ tells us at which point we have found them all. For $r = 2$, we just have to iterate the previous step over each vertex $v_1 \in \{v \in V(\Gamma) \mid d_\Gamma(v, v_0) \leq 1\}$, and so on.

This also shows that on a highly computable graph, the path-length metric $d_\Gamma : V(\Gamma)^2 \to \mathbb{N}$ is a computable function.

Finally, let us observe that a computable numbering of a computable group is automatically a computable numbering of $\mathrm{Cay}(G, S)$: two vertices $g, h \in G$ are joined if and only if $hg^{-1} \in S \cup S^{-1}$, and this is a decidable relation (for example, because the numbering makes the group operation computable and $S \cup S^{-1}$ is a finite set).

Proof of Proposition 2.4. We first prove the existence. Taking the usual order on $\mathbb{N}$, we define the decidable set $N \subset \mathbb{N}$ as follows. A natural number n lies in N if it is the minimal element of its equivalence class. Then $\nu : N \to \mathbb{N}/R$ is a numbering. The $\in$ relation in $\mathbb{N} \times \mathbb{N}^2/R$ is decidable. Given $n, m \in \mathbb{N}$, where m is a name for the equivalence class $\nu(m)$, we can decide if n lies in the equivalence class $\nu(m)$ by checking if $(n, m) \in R$.

We now prove the uniqueness, let ν and μ be numberings of $\mathbb{N}^2/R$ as in the statement. We see that the identity function from $(\mathbb{N}^2/R, \nu)$ to $(\mathbb{N}^2/R, \mu)$ is computable, the remaining direction holds for symmetry. Let n be a name for the equivalence class $\nu(n)$, we need to compute m such that $\mu(m) = \nu(n)$. As $\in$ is a decidable relation for ν, we can compute $r \in \mathbb{N}$ so that $r \in \nu(n)$. As $\in$ is also decidable for μ, we can compute the μ name m of the equivalence class where r lies. Thus $\mu(m) = \nu(n)$. $\square$

Proof of Proposition 2.5. Let $(G, \star)$ be a finitely generated group. If G is finite, then the statement holds. Indeed, it admits a computable numbering because a finite set $N \subset \mathbb{N}$ is always decidable and a function $f : N^2 \to N$ is always computable. Any two numberings are equivalent because any bijection between finite subsets of $\mathbb{N}$ is computable.

Let us now assume that G is infinite, that it has decidable word problem, and that S is a finite set of group generators. We already defined a numbering $\nu : \mathbb{N} \to G$ by means of the function $\pi : (S \cup S^{-1})^* \to G$, $w \mapsto w_G$. We now show that this is a computable numbering of G.

Indeed, let n, m be names for $g, h \in G$. We need to compute a name for gh. As the relation $\in$ is decidable for this numbering, we can compute two words $u, v \in (S \cup S^{-1})^*$ such that $u_G = g$ and $v_G = h$. Again, as $\in$ is decidable, we can compute a ν name for the equivalence class of $(uv)_G$. Thus ν is a computable numbering of G.

Now keep the numbering ν just defined, and assume that G has another computable numbering μ. Let us denote by $\bullet$ and $\diamond$ the computable group operations on $\mathbb{N}$ defined by $(\nu \times \nu)^{-1}(\star)$ and $(\mu \times \mu)^{-1}(\star)$, respectively. Let us write $S \cup S^{-1} = \{s_1, \dots, s_k\} \subset G$, and define $\{r_1, \dots, r_k\} \subset \mathbb{N}$ and $\{t_1, \dots, t_k\} \subset \mathbb{N}$ by $\mu(t_i) = \nu(r_i) = s_i$,where $i = 1, \dots, k$.

We show that the identity function is computable from (G, ν) to (G, ν'). On input $m \in \mathbb{N}$ (a ν name for a group element), compute natural numbers $i_1 \dots i_n$ such that $m = r_{i_1} \bullet \cdots \bullet r_{i_n}$. This can be done by searching exhaustively, and using that $\bullet$ is computable. Then output $t_{i_1} \diamond t_{i_2} \diamond \cdots \diamond t'_{i_n}$. Thus is a μ name for the same group element. □

The proof of Proposition 2.6 is left to the reader, as it is identical to the previous one.

3. Translation-like actions by $\mathbb{Z}$ on locally finite graphs

In this section we prove Theorem 1.3 and Theorem 1.5. For this we will work with finite and bi-infinite 3-paths.

Definition 3.1. A **3-path** (resp. **bi-infinite 3-path**) on a graph Γ is an injective function $f : [a, b] \to V(\Gamma)$, $[a, b] \subset \mathbb{Z}$, (resp. $f : \mathbb{Z} \to V(\Gamma)$) such that consecutive integers in the domain are mapped to vertices whose distance d_Γ is at most 3. It is called **Hamiltonian** if it visits all vertices.

It is known that a finite and connected graph admits a Hamiltonian 3-path, and we can pick the first and last of its vertices [19]. We start by showing that we can impose additional conditions on this 3-path.

Lemma 3.2. *Let Γ be a connected finite graph, and $u \neq v$ two of its vertices. Then Γ admits a Hamiltonian 3-path $f : [a, b] \to V(\Gamma)$ which starts at u, ends at v, and moreover satisfies the following two conditions.*

(1) If $b - a \geq 2$, then $d_\Gamma(f(a), f(a+1)) \leq 2$ and $d_\Gamma(f(b-1), f(b)) \leq 2$ (the first and last "jump" have length at most 2).
(2) There is no $c \in [a, b]$ with $c+1, c-1 \in [a, b]$ and such that $d_\Gamma(f(c-1), f(c)) = 3$ and $d_\Gamma(f(c), f(c+1)) = 3$ (there are no consecutive length 3 "jumps").

Using this we will construct bi-infinite 3-paths in an appropriate manner, and this will provide a proof of Theorem 1.5 and Theorem 1.3. Before proving the lemma, let us introduce some useful terminology.

Let Γ be a finite graph, and let $f : [a, b] \to V(\Gamma)$ be a 3-path on Γ. We say that f **starts** at $f(a)$ and **ends** at $f(b)$, that it goes from $f(a)$ to $f(b)$, and that $f(a), f(b)$ are the **endpoints** of f. We say that vertices in $f([a, b])$ are **visited** by f, and abreviate this set by $V(f)$.

Now let $f : [a, b] \to V(\Gamma)$ and $g : [c, d] \to V(\Gamma)$ be two 3-paths. We say that g **extends** f if it does it as a function, that is, its restriction to the domain of f is equal to f. We define the **concatenation** of f and g as the function h which visits the vertices visited by f and then the vertices visited by h, in the same order. More precisely, h has domain $[a, b + 1 + d - c]$, and is defined by

$$h(x) = \begin{cases} f(x) & x \in [a, b] \\ g(x - b - 1 + c) & x \in [b + 1, b + 1 + d - c]. \end{cases}$$

If $V(f) \cap V(g) = 0$ and $d_\Gamma(f(b), g(c)) \leq 3$, then h is a 3-path and extends f. Finally, the **inverse** of a 3-path $f : [a, b] \to V(\Gamma)$, denoted $-f$, is the 3-path with domain $[-b, -a]$ and which sends x to $f(-x)$.

Proof of Lemma 3.2. The proof is by induction on $|V(\Gamma)|$. The claim holds for $|V(\Gamma)| \leq 3$, as in that case any pair of vertices are at distance at most two.

Now let Γ be a connected finite graph with $|V(\Gamma)| \geq 3$, let $u \neq v \in V(\Gamma)$, and assume that the result holds for graphs with a strictly lower amount of vertices. We will show that there is a Hamiltonian 3-path on Γ from u to v as in the statement.

Let $\Gamma_1, \ldots, \Gamma_n$, $n \geq 1$ be the connected components of $\Gamma - v$, and assume that u lies in Γ_1. Moreover, let Γ_{n+1} be the graph whose only vertex is v.

For each $i \in [1, n + 1]$ we pick a pair of vertices u_i and v_i in Γ_i as follows. For $i = 1$ we pick $u_1 = u$. If Γ_1 has only one vertex then v_1 is also equal to u, otherwise v_1 is a vertex in Γ_1 adjacent to v. For $i \in [2, n]$ we pick u_i as a vertex in Γ_i adjacent to v. If Γ_i has only one vertex then we pick v_i equal to u_i, otherwise we take v_i as a vertex in Γ_i adjacent to u_i. Finally, we let u_{n+1} and v_{n+1} be equal to v. Thus for $i \in [1, n]$, v_i is at distance at most 3 from u_{i+1}.

We now invoke the inductive hypothesis on each Γ_i, $i \in [1, n + 1]$ (this is also correct when $|V(\Gamma_i)| = 1$), and obtain a 3-path f_i from u_i to v_i as in the statement. We define f as the concatenation of $f_1, \ldots, f_{n+1}$, and claim that f satisfies the required conditions.

First observe that f is a 3-path on Γ, as $V(f_i)$ and $V(f_j)$ are disjoint for $i \neq j \in [1, n + 1]$, and the last vertex visited by f_i is at distance at most 3 from the first vertex visited by f_{i+1}. It is clear that it visits all vertices exactly once, so it is Hamiltonian.

We now verify that f satisfies the first numbered condition in the statement. We need the following observation, which follows from the fact that u_{i+1} is at distance at most one from v.

Remark. If Γ_i is a singleton, $i \in [1, n]$, then $d(v_i, u_{i+1}) \leq 2$.

We claim that after $u = u_1$, f visits a vertex at distance at most 2 from u. Indeed, if $|V(\Gamma_1)| \geq 2$, then the claim holds by inductive hypothesis on f_1. Otherwise it follows from the fact that $u_1 = v_1$ and previous remark. Moreover, the last vertex visited by f_n, which is v_n, is at distance at most 2 from v ($n \geq 1$).

We now verify that f satisfies the second numbered condition in the statement. Take $c \in [a, b]$ such that $c + 1$ and $c - 1$ are both in $[a, b]$, we claim that some of $f(c - 1)$, $f(c + 1)$ is at distance at most 2 from $f(c)$. To prove this let Γ_i be the component containing $f(c)$, and note that $i \in [1, n]$. If Γ_i has one vertex, then $d_\Gamma(f(c), f(c + 1)) \leq 2$ by the remark above. Now assume that Γ_i has more than one vertex, and recall that f_i is a Hamiltonian 3-path on Γ_i from u_i to v_i as in the statement. If $f(c)$ is some of u_i or v_i, then the fact that f_i satisfies the first item in the statement shows the claim. If $f(c)$ is not one of u_i or v_i, then the fact that f_i satisfies the second item in the statement shows the claim. This finishes the proof. □

We now introduce the following condition. It will be used to extend 3-paths iteratively.

Definition 3.3. We say that a 3-path $f : [a, b] \to V(\Gamma)$ on a graph Γ is **bi-extensible** if the following conditions are satisfied.

(1) $\Gamma - f$ has no finite connected component.
(2) There is a vertex u in $\Gamma - f$ at distance at most 3 from $f(b)$.
(3) There is a vertex $v \neq u$ in $\Gamma - f$ at distance at most 3 from $f(a)$.

If only the two first conditions are satisfied, we say that f is **right-extensible**.

We first need to show that bi-extensible and right-extensible 3-paths exist. The proofs are easy, and are given by completeness.

Lemma 3.4. *Let Γ be an infinite, connected, and locally finite graph. Then for any pair of vertices u and v, there is a right-extensible 3-path which starts at u and visits v.*

Proof. Let f be a path joining u and v, which exists by connectedness. Now define Λ as the graph induced in Γ by the vertices in f and the ones in the finite connected components of $\Gamma - f$. Notice that as Γ is locally finite, there are finitely many such connected components, and thus Λ is a finite graph.

As Γ is connected there is some vertex w in Λ which is adjacent to some vertex outside Λ. By Lemma 3.2 there is a 3-path f' which is Hamiltonian on Λ, starts at u and ends in w. We claim that this f' is right-extensible. Indeed, our choice of Λ ensures that $\Gamma - f'$ has no finite connected component, and our choice of the endpoint w of f' ensures that the second condition in the definition of right extensible 3-path is satisfied. This finishes the proof. $\square$

Lemma 3.5. *Let Γ be an infinite, locally finite, connected graph, and let w be a vertex in Γ. Then there is a bi-extensible 3-path which visits w.*

Proof. Let $w' \neq w$ be adjacent to w, and define Λ to be the subgraph of Γ induced by the vertices w, w' and the ones in the finite connected components of $\Gamma - \{w, w'\}$. Thus Λ is a finite connected subgraph of Γ, has at least two vertices, and $\Gamma - \Lambda$ has no finite connected component.

As Γ is connected, there is a vertex u in Λ which is adjacent to some vertex u' in $\Gamma - \Lambda$. As $\Gamma - \Lambda$ has no finite connected component, there is another vertex v' in $\Gamma - \Lambda$ adjacent to u'. Finally as Λ is connected and has at least two vertices, there is a vertex v in Λ adjacent to u. Now we invoke Lemma 3.2 on Λ to obtain a 3-path f which is Hamiltonian and goes from u to v. We claim that f is bi-extensible. Indeed, our choice of Λ ensures that $\Gamma - f$ has no finite connected component. Moreover, u is at distance one from the vertex u' in $\Gamma - f$, and v is at distance at most 3 from the vertex $v \neq u$ in $\Gamma - f$. This finishes the proof. $\square$

We now show that under suitable conditions, a bi-extensible 3-path can indeed be extended to a larger bi-extensible 3-path, and we can choose the new 3-path to visit some vertex w. The second condition in Lemma 3.2 will be important in the following proof.

Lemma 3.6. *Let Γ be an infinite, locally finite, and connected graph. Let f be a bi-extensible 3-path on Γ, and let $u \neq v \in \Gamma - f$ be two vertices at distance at most 3 from the first and last vertex of f, respectively. If w is a vertex in the same connected component of $\Gamma - f$ that some of u or v, then there is a 3-path f' which extends f, is bi-extensible on Γ, and visits w. Moreover, we can assume that the domain of f' extends that of f in both directions.*

Proof. If u and v lie in different connected components of $\Gamma - f$, then then the claim is easily obtained by applying Lemma 3.4 on each of these components. Indeed, by Lemma 3.4 there are two right extensible 3-paths g and h in the corresponding connected components of $\Gamma - f$, such that g starts at u, h starts at v, and some of them visits w. Then the concatenation of $-g$, f and h satisfies the desired conditions.

We now consider the case where u and v lie in the same connected component of $\Gamma - f$, this graph will be denoted Λ. Thus Λ is infinite, locally finite, connected, contains u, v, and w.

We claim that there are two right-extensible 3-paths on Λ, g and h, satisfying the following list of conditions: g starts at u, h starts at v, some of them visits w, and $V(g) \cap V(h) = \emptyset$. In addition, $(\Lambda - g) - h$ has no finite connected component, and has two different vertices u' and v' such that u' is at distance at most 3 from the last vertex of g, and v' is at distance at most 3 from the last vertex of h.

Suppose that we have g, h as before. Then we can define a 3-path f' by concatenating $-g$, f and then h. It is easily seen from the properties above that then f' satisfies the conditions in the statement.

We now construct g and h. For this purpose, we take Λ' to be a connected finite subgraph of Λ which contains u, v, w and such that $\Lambda - \Lambda'$ has no finite connected component.

The graph Λ' can be obtained, for example, as follows. As Λ is connected, we can take a path f_u from u to w, and a path f_v from v to w. Then define Λ' as the graph induced by the vertices in $V(f_v)$, $V(f_u)$, and all vertices in the finite connected components of $(\Lambda - f_v) - f_u$. This graph has the desired properties by construction.

Let p be a Hamiltonian 3-path on Λ' from u to v as in Lemma 3.2, we will split p aproppiately to obtain g and h. Let w_1 be a vertex in Λ' which is adjacent to some vertex v' in $\Lambda - \Lambda'$, which exists as Λ is connected. Now we need the second condition in Lemma 3.2, which says that there is a vertex w_2 in Λ' whose distance from w_1 is at most 2, and such that p visits consecutively $\{w_1, w_2\}$. We will assume that p visits w_2 after visiting w_1, the other case being symmetric. As $\Lambda - \Lambda'$ has no finite connected component, there is a vertex u' in Λ which is outside Λ' and is adjacent to v'. Thus, w_1 is at distance at most 2 from u', and w_2 is at distance at most 3 from v'.

We define g and h by splitting p at the vertex w_1. More precisely, let $[a, c]$ be the domain of p, and let b be such that $p(b) = w_1$. Then h is defined as the restriction of p to $[a, b]$, and $-g$ is defined as the restriction of p to $[b+1, c]$. Thus h is a 3-path from v to w_1, and g is a 3-path from u to w_2. By our choice of Λ' and p, it is easily seen that the 3-paths h and g satisfy the mentioned list of conditions, and this finishes the proof. $\square$

Observe that when Γ has one or two ends, the hypothesis of Lemma 3.6 on u, v and w are always satisfied. We obtain a very simple and convenient statement: we can extend a bi-extensible 3-path so that it visits a vertex of our choice.

Corollary 3.7. *Let Γ be a locally finite, and connected graph with one or two ends. Let f be a bi-extensible 3-path, and let w be a vertex. Then f can be extended to a bi-extensible 3-path which visits w. Moreover we can assume that the domain of the new 3-path extends that of f in both directions.*

We are now in position to prove some results about bi-infinite 3-paths. We start with the Hamiltonian case, which is obtained by iteration of Corollary 3.7. When we deal with bi-infinite 3-paths, we use the same notation and abreviations introduced before for 3-paths, as long as they are well defined.

Proposition 3.8. *Let Γ be a one or two ended, connected, locally finite graph. Then it admits a bi-infinite Hamiltonian 3-path.*

Proof. Let $(v_i)_{i\in\mathbb{N}}$ be a numbering of the vertex set of Γ. We define a sequence of bi-extensible 3-paths $(f_i)_{i\in\mathbb{N}}$ on Γ in the following recursive manner. We define f_0 as a bi-extensible 3-path which visits v_0, this is possible by Lemma 3.6. Now assume that we have defined the bi-extensible 3-path f_i, and that it visits v_i. We define f_{i+1} as a bi-extensible 3-path on Γ which extends f_i, its domain extends the domain of f_i in both directions, and it visits v_i. This is possible by applying Corollary 3.7 to the graph Γ and the 3-path f_i.

Finally, we define a bi-infinite 3-path $f : \mathbb{Z} \to \Gamma$ by setting $f(n) = f_i(n)$, for i big enough. Then f is well defined because f_{i+1} extends f_i as a function, and the domains of f_i exhaust $\mathbb{Z}$. By construction f visits every vertex exactly once, so it is Hamiltonian. This finishes the proof. □

We now proceed with the non Hamiltonian case, where there are no restrictions on ends. We first prove that we can take a bi-infinite 3-path whose deletion leaves no finite connected component.

Lemma 3.9. *Let Γ be an infinite, connected, and locally finite graph. Then it admits a bi-infinite 3-path f such that $\Gamma - f$ has no finite connected component.*

Proof. By first applying Lemma 3.5 and then iterating Lemma 3.6, we obtain a sequence of bi-extensible 3-paths $(f_i)_{i\in\mathbb{N}}$ on Γ, such that f_{i+1} extends f_i for all $i \geq 0$, and such that their domains exhaust $\mathbb{Z}$. We define a bi-infinite 3-path $f : \mathbb{Z} \to \Gamma$ by setting $f(n) = f_i(n)$, for i big enough.

We claim that $\Gamma - f$ has no finite connected component. We argue by contradiction, let us assume that Γ_0 is a nonempty and finite connected component of $\Gamma - f$. Define V_1 as the set of vertices in Γ which are adjacent to some vertex in Γ_0, but which are not in Γ_0. Then V_1 is nonempty for otherwise Γ would not be connected, and is finite because Γ is locally finite. Thus there is a natural number k such that f_k has visited all vertices in V_1. By our choice of V_1, Γ_0 is a nonempty and finite connected component of $\Gamma - f_k$, and this contradicts that f_k is bi-extensible. □

Now the proof of the following result is by iteration of Lemma 3.9.

Proposition 3.10. *Let Γ an infinite, connected, and locally finite graph. Then there is a set of bi-infinite 3-paths $f_i : \mathbb{Z} \to \Gamma$, $i \in I$, such that $V(\Gamma) = \bigsqcup_{i\in I} V(f_i)$.*

Proof. As Γ is infinite, connected, and locally finite, we can apply Lemma 3.9 to obtain a bi-infinite 3-path f_0 such that $\Gamma - f_0$ has no finite connected component. Each connected component of $\Gamma - f_0$ is an infinite, connected, and locally finite graph, so we can apply Lemma 3.9 on each of them. Iterating this process in a tree-like manner, we obtain a family of 3-paths $f_i : \mathbb{Z} \to \Gamma$, $i \in I$. Observe that the countability of the vertex set ensures that $V(\Gamma) = \bigsqcup_{i\in I} V(f_i)$

□

We now derive Theorem 1.5 and Theorem 1.3 from this statements in terms of bi-infinite 3-paths.

Proof of Theorem 1.5. Let Γ be a graph as in the statement, and let $f_i, i \in I$ as in Proposition 3.10. We define $* : V(\Gamma) \times \mathbb{Z} \to V(\Gamma)$ by the expression

$$v * n = f(f^{-1}(v) + n),\ n \in \mathbb{N},$$

where f is the only f_i such that v is visited by f_i. Observe that $v * 1$ is well defined because $V(\Gamma) = \bigsqcup_{i\in I} V(f_i)$. This defines a translation-like action by $\mathbb{Z}$, where the distance from v to $v * 1$, $v \in V(\Gamma)$ is uniformly bounded by 3. □

Proof of Theorem 1.3. Let Γ be a graph as in the statement. By Proposition 3.8, Γ admits a Hamiltonian bi-infinite 3-path f. This defines a translation-like action by $\mathbb{Z}$ as in the proof of Theorem 1.5, and it is transitive because f is Hamiltonian.

We now prove that a locally finite graph which admits a transitive translation-like action by $\mathbb{Z}$ is connected and has one or two ends. This is stated in [27, Theorem 3.3] for graphs with uniformly bounded vertex degree, but the proof can be applied to locally finite graphs. We describe here another argument.

Let Γ be a locally finite graph which admits a transitive translation-like action by $\mathbb{Z}$. Connectedness is clear. It is also clear that Γ must be infinite, so it does not have zero ends. Assume now that it has at least 3 ends to obtain a contradiction. Denote this translation-like action by $*$, and let $J = \max\{d_\Gamma(v, v*1) \mid v \in V(\Gamma)\}$. Now take any vertex $v \in V(\Gamma)$, and define the function $f : \mathbb{Z} \to V(\Gamma)$ by $f(n) = v * n, n \in \mathbb{Z}$. This function is bijective and sends consecutive integers to vertices in Γ at distance d_Γ at most J.

As Γ has at least 3 ends, there is a finite set of vertices V_0 such that $\Gamma - V_0$ has at least three infinite connected components, which we denote Γ_1, Γ_2 and Γ_3. By enlarging V_0 if necessary, we can assume that any pair of vertices u, v which lie in different connected components in $\{\Gamma_1, \Gamma_2, \Gamma_3\}$ are at distance d_Γ at least $J + 1$. As V_0 is finite, there is a finite set $[n, m] \subset \mathbb{Z}$ such that $f([n, m])$ contains V_0. By our choice of V_0, n and m, the set of vertices $f[m + 1, \infty)$ is completely contained in one of Γ_1, Γ_2, or Γ_3. The same holds for $f(-\infty, n - 1]$, and thus the remaining infinite component is empty. This is a contradiction, and finishes the proof. □

We end this section by repeating a problem left in [27, Problem 3.5].

Problem 3.11. Find necessary and sufficient conditions for a graph to admit a transitive translation-like action by $\mathbb{Z}$.

Theorem 1.3 shows that the number of ends and connectedness completely classify locally finite graphs which admit a transitive translation-like action by $\mathbb{Z}$. The case of graphs with vertices of infinite degree may be different, as there is not a uniquely defined notion of ends there. This is discussed in detail in [8].

4. Computable translation-like actions by $\mathbb{Z}$

In this section we prove that every finitely generated group with decidable word problem admits a computable translation-like by $\mathbb{Z}$ with decidable orbit membership problem (Theorem 1.7). Let us recall here the proof scheme. For groups with at most two ends this is obtained from a computable version of Theorem 1.3. For the remaining case, we show that the group contains a subgroup isomorphic to $\mathbb{Z}$ with decidable subgroup membership problem.

The reader may observe that in the case where the group has two ends we provide two different proofs for Theorem 1.7. Indeed a group with two ends is virtually $\mathbb{Z}$, and it would be easy to give a more direct proof, but the intermediate statements have independent interest (Theorem 4.1 and Proposition 4.8).

4.1. **Computable and transitive translation-like actions.** In this subsection we prove the following result, and derive from it Theorem 1.7 for groups with one or two ends.

Theorem 4.1 (Computable Theorem 1.3)**.** *A highly computable graph which is connected and has one or two ends admits a computable and transitive translation-like action by $\mathbb{Z}$, where the distance between a vertex v and $v*1$ is uniformly bounded by 3.*

Joining this with Theorem 1.3 we obtain the following corollary.

Corollary 4.2. *A highly computable graph (resp. a finitely generated infinite group with decidable word problem) admits a transitive translation-like action by $\mathbb{Z}$ if and only if it admits a computable one.*

As discussed in the introduction, this shows great contrast between transitive translation-like actions and other infinite objects that arise in graph theory, at least from the perspective of computability theory.

We now proceed with the proof, which is divided in different lemmas. Recall that Theorem 1.3 was obtained by constructing a bi-infinite Hamiltonian 3-path (Proposition 3.8). In its proof we constructed a sequence of 3-paths $(f_i)_{i\in\mathbb{N}}$ satisfying certain conditions. We will show that these conditions are decidable, and this will suffice to compute the sequence by doing an exhaustive search. For this we will need to separate the one and two ended case. We start with the following result, which is used in both cases.

Lemma 4.3. *Let Γ be a highly computable graph which is connected. There is an algorithm which on input a finite set of vertices $V \subset V(\Gamma)$, halts if and only if $\Gamma - V$ has some finite conected component.*

Proof. The intuitive idea is that we can discover a finite connected component of $\Gamma - V$ by computing a finite but big enough subgraph. Let $v_0 \in \Gamma - V$, and for each $n \in \mathbb{N}$ let Γ_{n+1} denote the finite graph induced by all vertices v in $\Gamma - V$ with $d_\Gamma(v, v_0) \leq n + 1$. Observe that if some vertex v lies in Γ_{n+1} and its distance to v_0 is at most n, then we see all its $\Gamma - V$ edges or neighbors in the finite graph Γ_{n+1}. This if for some n_0 there is a finite connected subgraph of Γ_{n_0} which remains the same on Γ_{n_0+1}, then it will remain the same for all $n \geq n_0$, and thus it is a finite connected component of $\Gamma - V$. Thus the following algorithm proves the claim:

For each $n \in \mathbb{N}$, compute Γ_n and Γ_{n+1}, which is possible as Γ is highly computable, and do the following. Check if there is a finite connected component of Γ_n with no vertices in Γ_{n+1}, and halt if this is the case. □

We now can show that in the one ended case, the property of being a bi-extensible 3-path is decidable.

Lemma 4.4. *Let Γ be a highly computable multigraph with one end. There is an algorithm which on input a finite set of vertices $V \subset V(\Gamma)$, decides if $\Gamma - V$ has no finite connected component.*

In particular, it is decidable whether a 3-path is bi-extensible.

Proof. We prove the first claim. By Lemma 4.3, it remains to prove the existence of an algorithm which halts on input a finite set of vertices V if and only if $\Gamma - V$ is has no finite connected component. As Γ has one end, this is equivlent to ask if $\Gamma - V$ is connected. The following algorithm halts on input V, a finite set of vertices, if and only if $\Gamma - V$ is connected:

Compute the set V_0 of vertices in $\Gamma - V$ which are adjacent to some vertex in V. For each pair of vertices of V_0, search exhaustively for a path in $\Gamma - V$ which joins them. If at some point we have found such a path for every pair of vertices in V_0, halt.

We now observe that the second claim in the statement is obtained from the first. Note that the second and third condition in the definition of bi-extensible 3-path are clearly decidable. □

We now proceed with the two ended case, where we make use of the finite information contained in a finite set of vertices which disconnects the graph.

Lemma 4.5. *Let Γ be a highly computable multigraph with two ends, and let V' be a finite set of vertices such that $\Gamma - V'$ has two infinite connected components. There* *is an algorithm which on input a finite set of vertices $V \subset V(\Gamma)$ which contains V', decides if $\Gamma - V$ is has some finite connected component.*

Now let f_0 be a bi-extensible 3-path on Γ such that $\Gamma - f_0$ has two infinite connected components. It is decidable whether a 3-path f which extends f_0 is bi-extensible.

Proof. We now prove the first claim. By Lemma 4.3, it remains to show that there is an algorithm which halts on input V if and only if $\Gamma - V$ has no finite connected component. As V contains V', we know that $\Gamma - V$ has exactly two infinite connected components. Now the following algorithm proves the claim.

On input V, compute the set V_0 of vertices in $\Gamma - V$ which are adjacent to some vertex in V. For each pair of vertices of V_0, search exhaustively for a path in $\Gamma - V$ which joins them. Now halt if at some point we find enough paths so that V_0 can be written as $V_1 \sqcup V_2$, where every pair of vertices in V_i can be joined by a path in $\Gamma - V$, $i = 1, 2$

We now observe that the second claim in the statement is obtained from the first. Note that the second and third condition in the definition of bi-extensible 3-path are clearly decidable. □

We can now show tht the bi-infinite Hamiltonian path in Proposition 3.8 can be computed.

Proposition 4.6 (Computable Proposition 3.8)**.** *Let Γ be a highly computable graph satisfying the hypothesis in Theorem 1.3. Then it admits a Hamiltonian bi-infinite 3-path which is computable.*

Proof. Let $(v_i)_{i\in\mathbb{N}}$ the numbering associated to the highly computable graph Γ, so $V(\Gamma) = \{v_i |\ i \in \mathbb{N}\}$.

Now let f_0 be a 3-path which is bi-extensible and visits v_0. If Γ has two ends, then we also require that $\Gamma - f_0$ has two infinite connected components (in this case we do not claim that the path f_0 can be computed from a description of the graph, but it exists and can be specified with finite information).

Now define the sequence of 3-path $(f_i)_{i\in\mathbb{N}}$ in a recursive and computable manner. Assume that f_i has been defined, $i \geq 0$, and define f_{i+1} as any bi-extensible 3-path on Γ such that f_{i+1} extends f_i, its domain extends the domain of f_i in both directions, and it visits v_i. Its existence is guaranteed by applying Corollary 3.7 to the graph Γ and the 3-path f_i. Moreover, it can be computed by searching exhaustively, the conditions imposed on f_{i+1} are decidable thanks to Lemma 4.4 and Lemma 4.5.

The remaining of the proof is the same as before. We define $f : \mathbb{Z} \to \Gamma$ by $f(n) = f_i(n)$, for i big enough, and this is a and Hamiltonian 3-path on Γ. Moreover, it is computable becuse the sequence $(f_i)_{i\in\mathbb{N}}$ is computable. □

We can now prove Theorem 4.1.

Proof of Theorem 4.1. Let $f : \mathbb{Z} \to V(\Gamma)$ be a bi-infinite 3-path on Γ which is Hamiltonian and computable, which exists thanks to Proposition 4.6. It is enough to observe that the translation-like action $* : V(\Gamma) \times \mathbb{Z} \to V(\Gamma)$ defined by

$$v * n = f(f^{-1}(v) + n),\ n \in \mathbb{N}$$

is computable. □

As mentioned before, Theorem 4.1 implies immediately that Theorem 1.7 holds for groups with one or two ends.

Proof of Theorem 1.7 for groups with one or two ends. Let G be a finitely generated infinite group with one or two ends and decidable word problem, let $\Gamma =$

$\mathrm{Cay}(G,S)$ where S is a finite set of generators for G, and endow G with a computable numbering ν. This numbering makes Γ a highly computable graph. Now Theorem 4.1 yields a computable and transitive translation-like action on $(V(\Gamma), d_\Gamma)$, which is also a transitive and computable translation-like action on (G, d_S). □

4.2. **Computable normal forms from Stalling's theorem.** In this subsection we prove Theorem 1.7 for groups with two or more ends. For this we will show that a finitely generated group with two or more ends has a subgroup isomorphic to $\mathbb{Z}$ with decidable subgroup membership problem. This will be obtained from the computability of the normal form associated to Stalling's structure theorem (see Proposition 4.8 below).

For the reader's convenience, we recall some facts about HNN extensions, amalgamated products, and normal forms. We refer the interested reader to [20, Chapter IV]. Let $H = \langle S_H | R_H\rangle$, t a symbol not in S_H, and an isomorphism $\phi : A \to B$ between subgroups of H. The **HNN extension** relative to H and ϕ is the group with presentation

$$H*_\phi := \langle S_H, t |\ R_H, tat^{-1} = \phi(a), \forall a \in A\rangle.$$

Now let $T_A \subset H$ (respectively T_B) be a fixed set of representatives for equivalence classes of H modulo A (respectively B). A sequence of group elements $h_0, t^{\epsilon_1}, h_1, \dots, t^{\epsilon_n}, h_n$ is in **normal form** if $\epsilon_i \in \{1, -1\}$ and the following conditions are satisfied:

(1) $h_0 \in H$
(2) if $\epsilon_i = -1$, then $h_i \in T_A$
(3) if $\epsilon_i = 1$, then $h_i \in T_B$
(4) There is no subsequence of the form $t^\epsilon, 1_H, t^{-\epsilon}$.

For every $g \in H*_\phi$ there exists a unique sequence in normal form whose product equals g in $H*_\phi$. The situation is analogous for amalgamated products, which we define now. Consider two groups $H = \langle S_H |\ R_H\rangle$ and $K = \langle S_K | R_K\rangle$, and an isomorphism $\phi : A \to B$ between the subgroups $A \leqslant H$ and $B \leqslant K$. The **amalgamated product** of H and K relative to ϕ is the group with presentation

$$H *_\phi K = \langle S_H, S_K |\ a = \phi(a), \forall a \in A\rangle.$$

Let $T_A \subset H$ (respectively $T_B \subset K$) be a fixed set of representatives for H modulo A (respectively K modulo B). Then a sequence of group elements $c_0, c_1, \dots, c_n$ is in **normal form** if

(1) c_0 lies in A or B
(2) For $i \geq 1$, each c_i is in T_A or T_B
(3) For $i \geq 1$ $c_i \neq 1$
(4) successive c_i alternate between T_A and T_B

For each element $g \in H *_\phi K$, there exist a unique sequence in normal form whose product equals g in $H *_\phi K$.

In the terms previously defined, we have:

Theorem 4.7 (Stalling's structure theorem, [9])**.** *Let G be a finitely generated group with two or more ends. Then one of the following occurs:*

(1) G is an HNN extension $H*_\phi$.
(2) G is an amalgamated product $H *_\phi K$.

In both cases the corresponding isomorphism ϕ: $A \to B$ is between finite and proper subgroups A and B.

A classical application of normal forms is the following: if a group H has decidable word problem, and we take an HNN extension $H*_\phi$ satisfying some particular hypothesis, then the extension $H*_\phi$ has computable normal form, and as consequence, decidable word problem [20, pp 185]. For our application, we need a sort of converse of this result. The proof is direct, but we were unable to find this statement in the literature.

Theorem 4.8. *Let G be a finitely generated group with two or more ends and decidable word problem. Then the normal forms associated to the decomposition of G as HNN extension or amalgamated product is computable.*

Proof. Let us assume that we are in the first case, so G is (isomorphic to) an HNN extension of a group H:

$$H*_\phi := \langle S_H, t|\ R_H, tat^{-1} = \phi(a), a \in A\rangle.$$

Let us show that the normal forms of $H*_\phi$ described above is computable. For this purpose we will show that we can computably enumerate sequences of words $w_1, \ldots, w_n$ whose corresponding group elements $(w_1)_G, \ldots, (w_n)_G$ are in normal form (for fixed sets T_A and T_B), and be sure to enumerate normal forms for all group elements. To compute the normal form of a group element w_G given by a word w, we just enumerate such sequences $w_1, \ldots, w_n$ until we find one satisfying $w =_G w_1 \ldots w_n$. This is possible as G has decidable word problem.

First note that the fact that G is finitely generated and A is finite forces H to be finitely generated (see [7, page 35]), so we can assume that S_H is a finite set. Moreover H has decidable word problem because this property is inherited by subgroups.

Now we claim that $A = \{a_1, \ldots, a_n\}$ has decidable membership problem in H. Indeed, to decide if $w_H \in A$, we just have to check $w =_H a_i$ for $i = 1, \ldots, m$; this is possible as A is finite and the word problem of H is decidable. As a consequence of this, we can also decide if $u \in Av$ for any $u, v \in (S_H \cup S_H^{-1})^*$, as this is equivalent to decide if $uv_H^{-1} \in A$. The same is true for B.

Let us show how to enumerate a set of words $W_A \subset (S_H \cup S_H^{-1})^*$ whose corresponding group elements are a colection of representatives for H modulo A, namely T_A. First define u_0 be the empty word. Now assume that words $u_0, \ldots, u_n$ have been selected and search for a word $u_{n+1} \in (S_H \cup S_H^{-1})^*$ which is not in $Au_0, \ldots, Au_n$. It is clear then that W_A is a computably enumerable set of words, and that the set T_A of the group elements of H corresponding to these words is a set of representatives for H modulo A.

A set W_B corresponding to T_B can be enumerated analogously. Thus we can enumerate normal forms $w_1, \ldots, w_n$ as follows: w_1 is an arbitrary element of $(S_H \cup S_H^{-1})^*$, and the rest are words of W_A, W_B, or $\{t, t^{-1}\}$ so as to alternate as in the definition. This finishes the proof for the case of an HNN extension.

If G is an amalgamated product, the fact that G and C are finitely generated forces H and K to be finitely generated (see [7, page 43]), and thus we can assume that S_H and S_K are finite sets. The sets W_A and W_B corresponding to T_A and T_B can be computably enumerated in the same manner as above, and the rest of the argument is identical to the HNN case. □

Now the proof of the following result is inmediate.

Corollary 4.9. *Let G be a finitely generated group with two or more ends and decidable word problem. Then it has a subgroup isomorphic to $\mathbb{Z}$ with decidable subgroup membership problem.*

Proof. If G is an HNN extension, then the group $\langle t\rangle \leqslant G$ has decidable membership problem. Indeed a group element g is in this subgroup if and only if the normal form of g or g^{-1} is $1, t, 1 \ldots, t, 1$.

If G is an amalgamated product, let $u \in T_H$, $v \in T_K$ be any pair of non trivial elements. Then the subgroup $\langle uv\rangle \leqslant G$ is infinite cyclic and has decidable membership problem, as a group element g is in this subgroup if and only if the normal form of g or g^{-1} is $u, v, \ldots, u, v$. $\square$

As mentioned in the introduction, for translation-like actions coming from subgroups, the two properties (decidable orbit membership problem, and decidable subgroup membership problem) become equivalent.

Proposition 4.10. *Let G be a finitely generated group. Then a subgroup $H \leqslant G$ has decidable membership problem if and only if the action of H on G by right translations has decidable orbit membership problem.*

Proof. Let us denote by $*$ the right action $G \times H \to G$, $(g, h) \mapsto gh$. Now we just have to note that two elements $g_1, g_2 \in G$ lie in the same $*$ orbit if and only if ${g_1 g_2}^{-1} \in H$, and an element $g \in G$ lies in H if and only if it lies in the same $*$ orbit as 1_G.

It is clear how to rewrite this in terms of words, but we fill the details for completeness. The set $\{w \in (S \cup S^{-1})^* |\ w_G \in H\}$ equals the set of words $w \in (S \cup S^{-1})^*$ such that w and 1_G lie in the same orbit, which is a decidable set assuming that $*$ has decidable orbit membership problem. We now assume that $\{w \in (S \cup S^{-1})^* |\ w_G \in H\}$ is decidable. Given two words $u, v \in (S \cup S^{-1})^*$, they lie in the same $*$ orbit if and only if the word uv^{-1} lie in $\{w \in (S \cup S^{-1})^* |\ w_G \in H\}$. $\square$

We can now finish the proof of Theorem 1.7, where the only remaining step is to verify the computability of the action.

Proof of Theorem 1.7 for groups with two or more ends. Let G be a finitely generated infinite group with decidable word problem. By Corollary 4.9 there is an element $c \in G$ such that $\langle c \rangle$ is isomorphic to $\mathbb{Z}$, and has decidable subgroup membership problem in G. Thus the right action $* : G \times \mathbb{Z} \to G$, $(g, n) \mapsto (gc^n)$ has decidable orbit membership problem by Proposition 4.10.

It only remains to verify that this action is computable. This is obvious in terms of words, but we write the details for completeness. Let S be a finite set of generators for G, and let u be a word in $(S \cup S^{-1})^*$ such that $u_G = c$. Now, the function which given a word $w \in (S \cup S^{-1})^*$ and an integer $n \in \mathbb{Z}$ outputs the concatenation wu^n, is a computable function. Without loss of generality (Proposition 2.5) we can endow G with the numbering obtained from the surjection $(S \cup S^{-1})^* \to G, w \mapsto w_G$, and the previous paragraph makes it clear that the fuction $*$ is computable. $\square$

5. Medvedev degrees of effective subshifts

In this section we prove Theorem 1.8. For this we need some definitions. Let G be a finitely generated group, and A a finite set. A **subshift** is a subset $X \subset A^G$ which is closed in the prodiscrete topology, and invariant under the group action $G \curvearrowright A^G$ by left translations $(g * x)(h) \mapsto x(g^{-1}h)$. A **pattern** is a function $p : K \subset G \to A$ with finite domain, and it determines the **cylinder**

$$[p] = \{x \in A^G |\ x(k) = p(k) \quad \forall k \in K\}.$$

Cylinders are closed and open, and they form a basis for this topology. If $g * x \in [p]$ for some $g \in G$, we say that p **appears** on x.

A set of patterns $\mathcal{F}$ defines the subshift $X_{\mathcal{F}}$ of all elements $x \in A^G$ such that no pattern of $\mathcal{F}$ appears in x. We say that $X_{\mathcal{F}}$ is obtained by forbidding patterns of $\mathcal{F}$. Conversely, any subshift can be obtained by forbidding patterns.

A subshift X is called a **subshift of finite type** if it can be obtained by forbidding finitely many patterns. We now discuss a notion of effectiveness of subshifts, for which we need the following definition.

A **pattern coding** c is a finite set of tuples $\{(w_1, a_1), \ldots, (w_k, a_k)\}$, where $w_i \in S^*$ and $a_i \in A$, and it is called **consistent** if for each i, j, $w_i =_G w_j$ implies $a_i = a_j$. A consistent pattern coding as above can be associated to the pattern $p(c) : K \subset G \to A$, where K is the set of group elements associated to $\{w_1, \ldots, w_k\}$, and $p((w_i)_G) = a_i$.

A set of pattern codings $\mathcal{C}$ defines the subshift $X_{\mathcal{C}}$ of all elements $x \in A^G$ such that no pattern of the form $p(c)$ appears in x, where c ranges over $\mathcal{C}$. Thus inconsistent pattern codings are ignored by definition. A subshift X is called **effective** if $X = X_{\mathcal{C}}$ for a computably enumerable set of pattern codings $\mathcal{C}$.

The previous definition was introduced in [1] to define effective subshifts without assumptions on the word problem of the group. If we assume that G has decidable word problem, we can decide which pattern codings are consistent, and decide which ones corresponds to the same pattern. This gives a bijective numbering of the set of all patterns as in Proposition 2.4, and we can say that a set of patterns $\mathcal{F}$ is computably enumerable. It is then clear that a subshift X is effective if and only $X = X_{\mathcal{F}}$ for some computably enumerable set of patterns $\mathcal{F}$.

A third equivalent notion is that an effective subshift is a subshift which is also an effectively closed subset of A^G. For this to make sense, we need to define effectively closed subsets of A^G. In what follows we will transfer all computability notions from $A^{\mathbb{N}}$ to A^G using representations, in the sense of computability theory. This is used to define the Medvedev degree of a subset of A^G, and to show that this notion enjoys some stability properties.

5.1. **Computability on $A^{\mathbb{N}}$, and Medvedev degrees.** Despite $\mathbb{N}$ not being a group, we will use the same definitions of patterns and cylinders in $A^{\mathbb{N}}$. A word $w_0 \ldots w_n \in A^*$ can be identified with a pattern $\{0, \ldots, n\} \to A$, and thus $[w] = \{x \in A^{\mathbb{N}} | \; x_0 \ldots x_n = w\}$.

We review now some facts on computability on the Cantor space that will be needed, the reader is refered to [26]. The intuitive meaning in the following definition is the following. A function Φ on $A^{\mathbb{N}}$ is computable if there is a computable function on words which computes it via finite prefixes, that is, it takes as input a finite prefix of $x \in A^{\mathbb{N}}$ and outputs a finite prefix of $\Phi(x)$.

Definition 5.1 (Computable function)**.** A partial function $\Phi : D \subset A^{\mathbb{N}} \to B^{\mathbb{N}}$ is **computable** if there exists a partial computable function $\phi : A^* \to B^*$ which is compatible with the prefix order on words[3], and such that D is the set of sequences $x \in B^{\mathbb{N}}$ where the following holds. As n tends to infinity, $\phi(x_0 \ldots x_n)$ is always defined, its length tends to infinity, and $\bigcap_n [\phi(x_0 \ldots x_n)]$ is the singleton which only contains $\Phi(x)$.

An alternative definition uses the concept of *oracle*, which we have avoided to make this article more accesible[4].

Example 5.2. The shift function $\sigma : 2^{\mathbb{N}} \to 2^{\mathbb{N}}, \sigma x(n) := x(n+1)$ is computable. It is given by the function $s : 2^* \to 2^*$, $s(w_0 w_1 \ldots w_n) = w_1 \ldots w_n$.

The following is a fairly standard fact, but we give a proof as it will be used later.

[3] That is, if u is a prefix of v, then $\phi(u)$ is a prefix of $\phi(v)$.

[4] A reader familiar with the concept of *oracle* may prefer this definition: Φ is computable if there is a partial computable function with oracle $\phi : \mathbb{N} \to B$, such that with oracle x and input $n \in \mathbb{N}$ returns the n-th coordinate of $\Phi(x)$, that is $\phi^x(n) = \Phi(x)_n$ for $n \in \mathbb{N}$ and all x in the domain of Φ.

Proposition 5.3. *Let $f : \mathbb{N} \to \mathbb{N}$ be any computable bijection. Then the homeomorphism*

$$F : A^{\mathbb{N}} \to A^{\mathbb{N}}$$
$$x \mapsto x \circ f$$

is computable and with computable inverse $x \mapsto x \circ f^{-1}$.

Proof. Note that a computable bijection of $\mathbb{N}$ has computable inverse. Now we define the computable function ϕ as follows. On input $a_1 \dots a_n \in A^*$ compute $k = k(n)$ to be the biggest natural number such that $\{0, \dots, k\}$ is contained in $\{f(0), \dots, f(n)\}$, and then output $\phi(a_0 \dots a_n) = a_{f^{-1}(0)} \dots a_{f^{-1}(k)}$. Note that $\lim_{n\to\infty} k(n) = \infty$, and thus $\lim_{n\to\infty} |\phi(a_0 \dots a_n)| = \infty$, this shows that F is defined on all inputs. By construction we have that $F(x) = \cap_{n\in\mathbb{N}}[\phi(x_0 \dots x_n)]$. Thus we proved that F is computable.

Replacing f by f^{-1} in the argument, we see that the inverse of F, $x \to x \circ f^{-1}$ is also computable. □

Definition 5.4. Let $X \subset A^{\mathbb{N}}$ and $Y \subset B^{\mathbb{N}}$. The sets X, Y are **computably homeomorphic** if there exists partial computable functions $\Phi : A^{\mathbb{N}} \to B^{\mathbb{N}}$, $\Psi : B^{\mathbb{N}} \to A^{\mathbb{N}}$, such that X (resp Y) is contained in the domain of Φ (resp Ψ), and such that $\Psi(\Phi(x)) = x \quad \forall x \in X$.

Example 5.5. The sets $A^{\mathbb{N}}$ and $B^{\mathbb{N}}$ are computably homeomorphic for any A and B finite. Indeed, the usual homeomorphism between these sets is a computable function (for example, the one described in [16, Theorem 2-97]). A particularly simple case is when $A = \{0, 1, 2, 3\}$ and $B = \{0, 1\}$. A computable homeomorphism between $A^{\mathbb{N}}$ and $B^{\mathbb{N}}$ is given by the letter-to-word substitutions

$$0 \mapsto 00 \quad 1 \mapsto 01 \quad 2 \mapsto 10 \quad 3 \mapsto 11.$$

Definition 5.6. A subset $X \subset A^{\mathbb{N}}$ is **effectively closed**, denoted Π_1^0, if some of the following equivalent conditions holds.

(1) The complement of X can be written as $\bigcup_{w\in L}[w]$ for a computably enumerable set $L \subset A^*$.
(2) We can semi decide if a word $w \in A^*$ satisfies $[w] \cap X = \emptyset$.
(3) We can semi decide if a pattern $p : K \subset \mathbb{N} \to A$ satisfies $[p] \cap X = \emptyset$.

Example 5.7. Let $T \subset A^*$ be a tree, which means a set of words closed under prefix, and denote by $[T]$ the set of infinite paths of T, $\{x \in A^{\mathbb{N}} | \ \forall n \in \mathbb{N}, x_0 \dots x_n \in T\}$. If the tree T is computable (as subset of A^*), then its set of infinite paths $[T] = A^{\mathbb{N}} - \cup_{w \notin T}[w]$ is an effectively closed set. Conversely, it can be shown that every effectively closed set is the set of paths of a computable tree.

Definition 5.8 (Medvedev degrees)**.** Let $X \subset A^{\mathbb{N}}$, $Y \subset B^{\mathbb{N}}$. We say that Y is **Medvedev reducible** to X, written

$$Y \leq_{\mathfrak{M}} X,$$

if there is a partial computable function Φ defined on all elements of X, and such that $\Phi(X) \subset Y$. This is a preorder, and induces the equivalence relation $\equiv_{\mathfrak{M}}$ given by

$$X \equiv_{\mathfrak{M}} Y \iff X \leq_{\mathfrak{M}} Y \text{ and } Y \leq_{\mathfrak{M}} X.$$

Equivalence classes of $\equiv_{\mathfrak{M}}$ are called **Medvedev degrees**. Medvedev degrees form a lattice with interesting properties, a survey on the subject is [14]. If we regard a set $X \subset A^{\mathbb{N}}$ as the set of *solutions* to a problem, then the Medvedev degree of X, $\deg_{\mathfrak{M}}(X)$, measures how hard is it to construct a solution, where hard means hard to compute. From the definition, one can see that:

(1) If X has a computable point x_c, then it is minimal for $\leq_{\mathfrak{M}}$. This is proved by noting that the constant function $\Phi : A^{\mathbb{N}} \to X$, $x \mapsto x_c$ is computable. The Medvedev degree of this set is denoted $0_{\mathfrak{M}}$.
(2) The empty set $\emptyset$ is maximal for $\leq_{\mathfrak{M}}$. This reflects that for $\leq_{\mathfrak{M}}$, the hardest problem is one with no solutions.
(3) If X and Y are computably homeomorphic, then $X \equiv_{\mathfrak{M}} Y$.

As $2^{\mathbb{N}}$ and $A^{\mathbb{N}}$ are computably homeomorphic for any finite set A, it is enough to consider Medvedev degrees of subsets of $2^{\mathbb{N}}$.

An important sublattice of the lattice of Medvedev degrees is that of Medvedev degrees of effectively closed subsets. This is a countable class, its elements admit a finite description, and it exhibits many interesting properties. Natural and geometrical examples of effectively closed sets are subshifts of finite type, and more generally effective subshifts. As mentioned in the introduction, it is known that all Π_1^0 Medvedev degrees can be attained by two dimensional subshifts of finite type [28], and one dimensional effective subshifts [22].

5.2. **Computability on A^G.** In this subsection we define the Medvedev degree of a subset of A^G using representations. For the definition and its stability properties it will be essential to assume that G is a finitely generated infinite group with decidable word problem.

Representations are the uncountable version of numberings, as defined in 2.4. We recall the following definitions from [4, Chapter 9]. A **represented space** is a pair (X, δ) where X is a set and δ is a **representation** of X, that is, a partial surjection $\delta : \mathrm{dom}(\delta) \subset A^{\mathbb{N}} \to X$. A representation allows us to transfer the computability notions from $A^{\mathbb{N}}$ to X. For example, in a represented space (X, δ), a subset $Y \subset X$ is **effectively closed** when $\delta^{-1}(Y) \subset A^{\mathbb{N}}$ is an effectively closed set. Moreover, if $(X', \delta' : A'^{\mathbb{N}} \to X')$ is another represented space, a function $F : X \to X'$ is **computable** when $\delta'^{-1} \circ F \circ \delta : A^{\mathbb{N}} \to A'^{\mathbb{N}}$ is a computable function. Finally, two representations of X, $\delta : A^{\mathbb{N}} \to X$ and $\delta' : B^{\mathbb{N}} \to X$, are called **equivalent** if the identity function $X \to X$ is computable between (X, δ) and (X, δ'). In this case, both representations induce the same computability notions on X.

We will consider the following representation, which is also a total function and a homeomorphism.

Definition 5.9. Let G be a finitely generated infinite group with decidable word problem, and ν a computable numbering of G. We define the representation δ by

$$\begin{aligned} \delta : A^{\mathbb{N}} &\to A^G \\ x &\mapsto x \circ \nu^{-1}. \end{aligned}$$

Recall from Section 2 that a group as in the statement admits a computable numbering. The next proposition shows that the computability notions that we obtain on A^G do not depend on the choice of ν.

Proposition 5.10. *In the previous definition, any two computable numberings induce equivalent representations.*

Proof. Let $F : A^G \to A^G$ be the identity function. Then $\delta'^{-1} \circ F \circ \delta : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is given by $x \mapsto x \circ \nu^{-1} \circ \nu'$. This is a computable function by Proposition 5.3 and the fact that $\nu^{-1} \circ \nu' : \mathbb{N} \to \mathbb{N}$ is a computable bijection of $\mathbb{N}$. □

Indeed, computability notions on A^G are also preserved by group isomorphisms.

Proposition 5.11. *Consider a group G' and a representation for $A^{G'}$ as in Definition 5.9. If $f : G \to G'$ is a group isomorphism, then the associated function $F : A^{G'} \to A^G$, $x \mapsto x \circ f$ is computable.*

The proof is omitted, as it is identical to the previous one by applying Proposition 2.6. This means that the computability notions being considered on A^G are preserved if we rename group elements (for example, by taking different presentations of the same group).

We can now define the Medvedev degree of a subset of A^G.

Definition 5.12. For a subset $X \subset A^G$, we define $\deg_{\mathfrak{M}} X = \deg_{\mathfrak{M}}(\delta^{-1}X)$.

This definition does not depend on δ, as long as δ comes from a computable numbering of G. Let us now review some basic facts about effectively closed sets on A^G. With this representation we recover the familiar description of effectively closed subsets in terms of cylinders.

Proposition 5.13. *A subset $X \subset A^G$ is effectively closed if and only if we can semi decide if a pattern $p : K \subset G \to A$ satisfies $[p] \cap X = \emptyset$.*

Proof. Given a pattern $p : K \subset G \to A$, we just have to compute a pattern $p' : K \subset \mathbb{N} \to A$ such that $p = p' \circ \nu$. Then $[p] \cap X = \emptyset$ if and only if $[p'] \cap \delta^{-1}(X) = \emptyset$, which can be semi decided as $\delta^{-1}(X)$ is effectively closed. $\square$

In [1, Lemma 2.3] it is shown that for recursively presented group and in particular one with decidable word problem, an effective subshift has a maximal -for inclusion-computably enumerable set of pattern codings associated to forbidden patterns. In other words, the set of all patterns p with $[p] \cap X = \emptyset$ is computably enumerable. Joining this with Proposition 5.13, we obtain:

Proposition 5.14. *A subshift $X \subset A^G$ is effective if and only if it is an effectively closed subset of A^G.*

5.3. **The subshift of translation-like actions, and the main construction.** In this subsection we describe how to code translation-like actions as a subshift, and then we use this to prove Theorem 1.8. Let G be an infinite group, S a finite set of generators, and $J \in \mathbb{N}$ (we will not assume yet that G has decidable word problem).

Definition 5.15. Let $T_J(\mathbb{Z}, G)$ be the set of all translation-like actions $* : \mathbb{Z} \times G \to G$ such that $\{d_S(g, g * 1) |\ g \in G\}$ is bounded by J.

Recall that $B(1_G, J)$ is the ball $\{g \in G \mid d_S(g, 1_G) \leq J\}$. We will consider the finite alphabet $B = B(1_G, J) \times B(1_G, J)$. Informally, a configuration $x \in B^G$ can be thought of as having an incoming and outogoing arrow at each $g \in G$. If $x(g) = (l, r)$, we can think that g has an outgoing arrow to gr, and an incoming arrow from gl, see Figure 1 on page 24.

Any translation-like action $* \in T_J(\mathbb{Z}, G)$ defines the configuration $x_* \in B^G$ by the condition

$$\forall g \in G \quad x_*(g) = (l, r) \iff g * -1 = gl \text{ and } g * 1 = gr.$$

Definition 5.16. Let $X_J(\mathbb{Z}, G)$ be the set of all x_*, where $*$ ranges over $T_J(\mathbb{Z}, G)$.

We will prove that $X_J(\mathbb{Z}, G)$ is a subshift, but we first introduce some notation. L and R stand for the projections $B \to B(1_G, J)$ to the left and right coordinate, respectively. Now an arbitrary element $x \in B^G$ defines two semigroup actions as follows. For $m \in \mathbb{Z}_{\geq 0}$ and $g \in G$, define the element $g *_x m$, by declaring $g *_x 0 = g$, $g *_x 1 = g \cdot R(x(g))$, and $g *_x (m+1) = (g *_x m) *_x 1$. For $m \in \mathbb{Z}_{\leq 0}$, we define $g *_x m$ by $g *_x -1 = g \cdot L(x(g))$ and $g *_x (m-1) = (g *_x m) *_x -1$. If $p : K \subset G \to B$ is a pattern, we give $g *_p m$ the same meaning as before, as long as it is defined.

It is easily seen that for arbitrary $x \in B^G$, we have

$$(g *_x n) *_x m = g *_x (n + m)$$

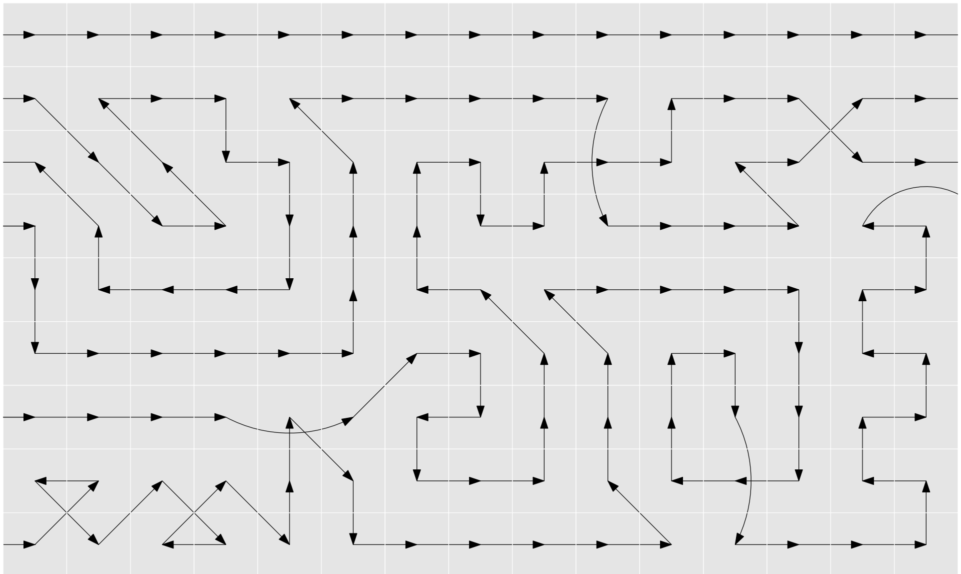

FIGURE 1. Representation of some orbits of a translation-like action in $T_2(\mathbb{Z},\mathbb{Z}^2)$, or alternatively, a finite piece of configuration in $X_2(\mathbb{Z},\mathbb{Z}^2)$. In this case, $\mathbb{Z}^2$ is endowed with the set of four generators $S = \{(\pm 1, 0), (0, \pm 1)\}$.

for both n, m positive or both negative integers. It may fail for $n, m \in \mathbb{Z}$, but this is easily fixed with local rules.

Proposition 5.17. *The set $X_J(\mathbb{Z}, G)$ is a subshift. If we assume that G has decidable word problem, it is an effective subshift.*

Proof. We claim that $X_J(\mathbb{Z}, G) = X_{\mathcal{J}}$, where $\mathcal{J}$ is the set of all patterns $p : B(1_G, n) \to B$, $n \in \mathbb{N}$, such that some of the following fails

(1) $(1_G *_p 1) *_p -1 = 1_G$, $(1_G *_p -1) *_p 1 = 1_G$
(2) For any nonzero $m \in \mathbb{Z}$, $1_G *_p m \neq 1_G$.

If $* \in T_J(\mathbb{Z}, G)$, it is clear that no pattern of $\mathcal{J}$ may appear on x_*, by the definition of action and translation-like action. Thus $X_J(\mathbb{Z}, G) \subset X_{\mathcal{J}}$. Now let $x \in X_{\mathcal{J}}$. We claim that $*_x$ is a translation-like action. To see that it is a group action, first note that for $g \in G$, $g *_x 0 = g$ by definition. An easy induction on $\max\{|n|, |m|\}$ shows that $(g *_x n) *_x m = g *_x (n + m)$ for any $n, m \in \mathbb{Z}$ and $g \in G$. The group action $*_x$ is free because of the second condition, and the boundedness condition comes from the alphabet chosen. Thus $*_x$ is a translation-like action.

To show that x lies in $X_J(\mathbb{Z}, G)$, we have to check that it is of the form x_* for some translation-like action $*$, and it is clear that $x = x_{(*_x)}$. Thus $X_J(\mathbb{Z}, G)$ is a subshift.

Now let us assume that G has decidable word problem. Note that the definition of $*_p$ above is recursive. Given an arbitrary pattern p, and $m \in \mathbb{Z}$, we can decide if the group element $1_G *_p m$ is defined, and compute it. This shows that the conditions (1) and (2) are decidable over patterns, and thus $\mathcal{J}$ is a decidable set. This shows that $X_{\mathcal{J}}$ is an effective subshift. □

We now describe a subshift on G whose elements describe, simultaneously, translation-like actions, and configurations from a subshift over $\mathbb{Z}$. Let A be an arbitrary finite alphabet, and let B be the alphabet already defined and which depends on the natural number J. Note that elements of $(A \times B)^G$ can be conveniently written as (y, x) for $y \in A^G$ and $x \in B^G$; we will write $\pi_A : A \times B \to A$ and $\pi_B : A \times B \to B$ for the projections to the first and second coordinate, respectively.

Definition 5.18. For a one dimensional subshift $Y \subset A^{\mathbb{Z}}$, let $Y[X_J(\mathbb{Z}, G)]$ be the set of all configurations $(y, x) \in (A \times B)^G$ such that

(1) $x \in X_J(\mathbb{Z}, G)$, and
(2) for any $g \in G$, the $A^{\mathbb{Z}}$ element defined by $y(m) = \pi_A(y(g *_x m)$ lies in Y.

Proposition 5.19. *The set $Y[X_J(\mathbb{Z}, G)]$ is a subshift. If we assume that G has decidable word problem and Y is an effective subshift, then $Y[X_J(\mathbb{Z}, G)]$ is an effective subshift.*

Proof. Let $\mathcal{F}$ be the set of all patterns in $\mathbb{Z}$ that do not occur in X, so that $X = X_{\mathcal{F}}$, and let $\mathcal{J}$ as before, so that $X_{\mathcal{J}} = X_J(\mathbb{Z}, G)$. Define $\mathcal{H}$ to be the set of all patterns $p : B(1_G, n) \to A \times B$, $n \in \mathbb{N}$, such that one of the following occurs. Denote $q = \pi_B \circ p : B(1_G, n) \to B$.

(1) The pattern q lies in $\mathcal{J}$.
(2) For some $m \in \mathbb{N}$ the elements $g *_q 1, \ldots, g *_q m$ are all defined, lie in $B(1_G, n)$, and the pattern $r : \{1, \ldots, m\} \subset \mathbb{Z} \to A, r(k) = \pi_A(g \underset{q}{*} k)$ lies in $\mathcal{F}$.

As before, it is a rutinary verification that $x \in Y[X_J(\mathbb{Z}, G)]$ if and only if $x \in X_{\mathcal{H}}$.

Now assume that G has decidable word problem. Let us argue that $\mathcal{H}$ is a computably enumerable set, provided that the same holds for $\mathcal{F}$. Given a pattern p, we can compute the pattern q. The first condition is decidable, as we already proved that $\mathcal{J}$ is a decidable set. For the second, note that given p and $m \in \mathbb{N}$, we can also compute the pattern r. As $\mathcal{F}$ is a computable enumerable set, we can semi decide that r is in $\mathcal{F}$. This shows that $\mathcal{H}$ is a computably enumerable set, and finishes the proof. $\square$

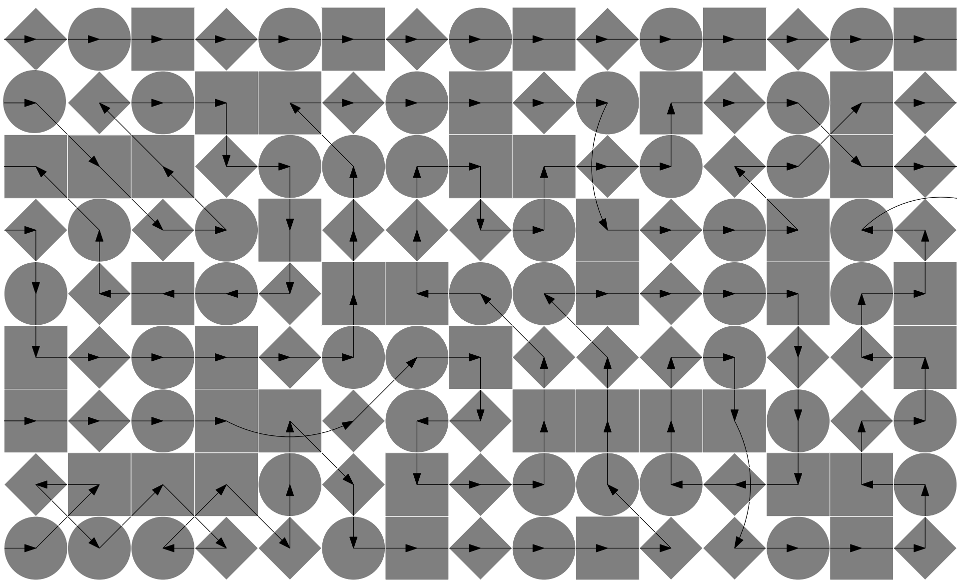

FIGURE 2. A finite piece of configuration in $Y[X_2(\mathbb{Z}, \mathbb{Z}^2)]$, where A is the alphabet containing the symbols of a circle, square, and rhombus, and $Y \subset A^{\mathbb{Z}}$ is the orbit closure of the periodic sequence that repeats circle, square, and a rhombus in that order.

The previous construction still makes sense if we replace $\mathbb{Z}$ by another finitely generated group H. In this case, the alphabet B depends on the generators of H. This is described in detail in [3, 18]. We will write $X_J(H,G)$ and $Y[X_J(\mathbb{Z},G)]$ with the same meaning as before, but only for reference reasons. It is natural to ask what properties are preserved by the map

$$Y \subset A^H \mapsto Y[X_J(H,G)] \subset (A\times B)^G.$$

The following is known.

(1) In [18], E. Jeandel proved that when H is a finitely presented group, the map above preserves the property of being a weakly aperiodic subshift, and of being empty or nonempty. This shows the undecidability of the emptiness problem for subshifts of finite type, and the existence of weakly aperiodic subshifts on new groups.

(2) In [3], S. Barbieri proved that when H and G are amenable groups, the topological entropy satisfies

$$h(Y[X]) = h(Y) + h(X).$$

This construction is used to classify the entropy of subshifts of finite type on some amenable groups.

In the present paper we use the previous construction because it preserves the constructive complexity of a subshift. We already proved that it preserves the property of being an effective subshift, which is folklore. In the following result we use Theorem 1.7 to show that this construction also preserves the Medvedev degree of a subshift Y for $H=\mathbb{Z}$, and J big enough. This shows Theorem 1.8, as the same classification for $H=\mathbb{Z}$ was proved by J. Miller in [22].

Theorem 5.20. *Let G be a finitely generated infinite group with decidable word problem, and let $J\in\mathbb{N}$ such that G admits a translation-like action by $\mathbb{Z}$ with decidable orbit membership problem. Then for any $Y\in A^{\mathbb{Z}}$,*

$$Y \equiv_{\mathfrak{M}} Y[X_J(\mathbb{Z},G)].$$

Proof. Recall that the inequality $Y \geq_{\mathfrak{M}} X$ holds if there exists a computable function Φ with $\Phi(Y)\subset X$, and this means that we can compute elements of X using elements of Y using a single algorithm. In this case, Φ is a computable function between the represented spaces $A^{\mathbb{Z}}$ and $(A\times B)^G$.

The obvious inequality is $Y[X_J(\mathbb{Z},G)] \geq_{\mathfrak{M}} Y$, that is, we can compute an Y element using a $Y[X_J(\mathbb{Z},G)]$ element. Informally, on input (y,x) we can just follow the arrows from 1_G and read the A component of the alphabet. This outputs an element of Y by definition.

Formally, we define the function $\Phi : Y[X_J(\mathbb{Z},G)] \to Y, (y,x)\mapsto z$ by

$$z(n) = y(\pi_A(1_G * n)), n\in\mathbb{Z}.$$

It is clear from the expression above how to compute z on input (y,x), and thus, Φ is computable.

For the remaining inequality, we need to compute a $Y[X_J(\mathbb{Z},G)]$ element using an element $z\in Y$. For this, let $*$ be a translation-like action as in Theorem 1.7.

The intuitive idea is to define $(y,x)\in Y[X_J(\mathbb{Z},G)]$ by leting $x=x_*$, which is a computabe point of B^G as $*$ is a computable function. Then we define y by coloring each orbit of $*$ with the sequence z. For this purpose, we use that $*$ has decidable orbit membership problem to compute a sequence of representatives for each orbit, and then we overlay z on each one of these orbits starting from the chosen representative.

Let $(g_n)_{n\in\mathbb{N}}$ be a computable numbering of G. We compute a sequence of representatives for orbits of $*$ as follows. Define the computable sequence $(k_i)_{i\in\mathbb{N}}$ by setting $k_0 = 0$, and defining k_{i+1} as the minimal natural number such that $g_{k_{i+1}}$ lies in a different orbit by $*$ than any of $g_{k_0}, \dots, g_{k_i}$. Note that this can be decided because $\{g_{k_0}, \dots, g_{k_i}\}$ is a finite set and $*$ has decidable orbit membership problem. Thus the sequence $(g_{k_i})_{i\in\mathbb{N}}$ is computable and contains exactly one element in each orbit of $*$.

We now define a computable function $\Psi_A : A^{\mathbb{Z}} \to A^G$ as follows. On input z it outputs the element y defined by

$$y(g_{k_i} * n) = z(n), \quad i \in \mathbb{N},\ n \in \mathbb{Z}.$$

This defines $y(g)$ for all $g \in G$, as $(g_{k_i})_{i\in\mathbb{N}}$ contains exactly one element in each orbit of $*$.

To see that Ψ_A is a computable function, note that given any $g \in G$ we can first compute k_i such that g lies in the same orbit as g_{k_j} (as the sequence $(k_i)_{i\in\mathbb{N}}$ is computable), and then use that the fact that the action $*$ is computable to find $n \in \mathbb{N}$ satisfying $g = g_{k_i} * n$.

Now define $\Psi_B : A^{\mathbb{Z}} \to B^G$ as the constant function x_*, which is computable as x_* is a computable point.

The previous two steps show that the function $\Psi : A^{\mathbb{Z}} \to (A \times B)^G$ defined by $z \mapsto \Psi(z) = (\Psi_A(z), \Psi_B(z))$ is computable. It satisfies $\Psi(Y) \subset Y[X_J(\mathbb{Z}, G)]$ by construction, and this finishes the proof. $\square$

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