INFINITE EULERIAN TRAILS ARE COMPUTABLE ON GRAPHS WITH VERTICES OF INFINITE DEGREE

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Abstract. The Erdős, Grünwald and Weiszfeld theorem provides a characterization of infinite graphs which are Eulerian. That is, infinite graphs which admit infinite Eulerian trails. In this article we complement this theorem with a characterization of those finite trails that can be extended to infinite Eulerian trails. This allows us to prove an effective version of the Erdős, Grünwald and Weiszfeld theorem for a class of graphs that includes non locally finite ones, generalizing a theorem of D.Bean.

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1. Introduction

A basic notion in graph theory is that of an Eulerian trail, that is, a trail which visits every edge exactly once. It is very easy to determine whether a finite graph admits an Eulerian trail or Eulerian closed trail: by Euler's theorem we just need to check the parity of the vertex degrees. Other classic problems in graph theory such as the existence of Hamiltonian paths or the existence of 3-colorings do not have such a simple answer. This has important consequences in complexity theory, where many algorithmic problems are reduced to graph theoretic problems.

The Erdős, Grünwald and Weiszfeld theorem generalizes Euler's theorem to infinite graphs. This result is a characterization of which infinite graphs admit

2020 *Mathematics Subject Classification.* 05C63, 03D55, 05C45, 68R10, 03D99, 68Q01.
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infinite Eulerian trails [6]. According to this result the existence of infinite Eulerian trails on a graph is determined by the parity of the vertex degrees, and a topological notion called *ends*. For example, a graph which looks like an infinite line in one direction has one end, a two-sided infinite line has two ends, and the rooted infinite binary tree has infinitely many ends.

The Erdős, Grünwald and Weiszfeld theorem has been a source of interest in effective combinatorics that has been revisited in recent years [15, 2, 13, 14], being one of the few results in the theory of infinite graphs with an effective counterpart. The fundamental result was proved by D.Bean, who showed that this result is *effective* for locally finite graphs. This means that it is possible to compute an infinite Eulerian trail on a locally finite graph by looking at finite portions of the graph.

The formal statement considers highly computable graphs, that is, computable graphs where the vertex degree function is computable. In simple words this means a locally finite graph for which an algorithm is capable of drawing finite portions of any desired size.

Bean's theorem asserts that a highly computable graph admits an infinite Eulerian trail if and only if it admits a computable one [2]. This exhibits a stark contrast with other fundamental results in the theory of infinite graphs which are not effective for highly computable graphs, including König's infinity Lemma, Hall's matching theorem [16], Ramsey's theorem [18, 11], results regarding edge and vertex colorings [1, 17], and more recently, domatic partitions [12].

Our results require rather long definitions, so we provide a brief account of them before going into details. Our first main result complements the Erdős, Grünwald and Weiszfeld theorem by characterizing those trails which can be extended to infinite Eulerian trails. This is a purely graph theoretical result with no computability involved that might be of independent interest, and that constitutes the main technical contribution of this paper. As a consequence, we will obtain an effective version of the Erdős, Grünwald and Weiszfeld theorem beyond locally finite graphs.

In our results we consider computable graphs for which the vertex degree function

$$V(G) \to \mathbb{N} \cup \{\infty\}$$

is computable. We call these *moderately computable graphs*. For these graphs we prove that it is algorithmically decidable whether a finite trail can be extended to an infinite Eulerian trail. This result is naturally translated into a computability property of the space of infinite Eulerian trails in the context of computable analysis, namely, that it is both effectively closed and co-effectively closed. In particular, it follows that one can uniformly compute a collection of infinite Eurelian trails that is dense in this space, thus generalizing (by allowing non locally finite graphs) and strengthening (by computing a density of such trails) Bean's theorem.

To our knowledge, this is the first positive computability result for infinite graphs where vertices of infinite degree are allowed. We also mention that we do not restrict ourselves to simple graphs, that is, the graphs under consideration may have loops and multiple edges between vertices. For example, our results apply to the graph with a single vertex and infinitely many loops.

When considering infinite Eulerian trails, we need to distinguish between one-sided infinite Eulerian trails, whose edge set indexed by $\mathbb{N}$, and two-sided infinite Eulerian trails, whose edge set is indexed by $\mathbb{Z}$. The previous discussion is valid for both classes of trails. In what follows we introduce some definitions which are needed to state our results precisely, and which will be used along the whole document.

2. Definitions and results

We start with the concept of ends, defined for topological spaces in [9, 7], for graphs in [8], and indeed anticipated in the characterization proved by Erdős, Grünwald and Weiszfeld. We shall define the **number of ends** of a graph G as the supremum of the number of infinite connected components of the graph $G - E$ where E ranges over all finite sets of edges of G, and where $G - E$ denotes the subgraph of G induced by all edges in the edge set $E(G)$ of G which are not in E.

In modern terms, the Erdős, Grünwald and Weiszfeld theorem and the conditions involved can be stated as follows.

Theorem 2.1 ([6])**.** *A graph G admits a one-sided (resp. two-sided) infinite Eulerian trail if and only if it satisfies $\mathscr{E}_1$ (resp. $\mathscr{E}_2$).*

Definition 2.2. $\mathscr{E}_1$ stands for the following set of conditions for a graph G.

- $E(G)$ is countable, infinite, and G is connected.
- Either G has exactly one vertex with odd degree, or G has at least one vertex with infinite degree and no vertices with odd degree.
- G has one end.

Definition 2.3. $\mathscr{E}_2$ stands for the following set of conditions for a graph G.

- $E(G)$ is countable, infinite, and G is connected.
- The degree of each vertex is infinite or even.
- G has one or two ends. Moreover, if E is a finite set of edges which induces a subgraph where all vertices have even degree, then $G - E$ has one infinite connected component.

As we explained before, in this article we complement Theorem 2.1 by characterizing, in a graph satisfying the corresponding hypotheses, those trails that can be extended to one-sided or two-sided infinite Eulerian trails. These conditions, which we call right-extensible and bi-extensible, turn out to be as simple as possible. By this we mean that they are obviously satisfied by a trail which can be extended to a larger trail, and in particular to an infinite Eulerian trail. In order to state them precisely, we also need the notion of distinguished vertex.

Definition 2.4. Let G be a graph satisfying $\mathscr{E}_1$. The set of **distinguished** vertices of G is the following. If G has one vertex with odd degree, then this is its only distinguished vertex. If G does not have one vertex with odd degree, then all vertices with infinite degree in G are distinguished.

We denote by $G - t$ the subgraph of G induced by the set of all edges in G which are not visited by t.

Definition 2.5. Let G be a graph satisfying $\mathscr{E}_1$. We say that a trail t is **right-extensible** in G if the following conditions hold.

(1) $G - t$ is connected.
(2) The initial vertex of t is distinguished in G.
(3) There is an edge e incident to the final vertex of t which was not visited by t.

Definition 2.6. Let G be a graph satisfying $\mathscr{E}_2$. We say that a trail t is **bi-extensible** in G if the following conditions hold.

(1) $G - t$ has no finite connected components.
(2) There is an edge e incident to the final vertex of t which was not visited by t.
(3) There is an edge $f \neq e$ incident to the initial vertex of t which was not visited by t.

We can now state our main result regarding the extension of finite trails to infinite Eulerian trails.

Theorem A. *In a graph satisfying $\mathscr{E}_1$:*

- *A vertex is the initial vertex of a one-sided infinite Eulerian trail if and only if it is distinguished.*
- *A trail can be extended to a one-sided infinite Eulerian trail if and only if it is right-extensible.*

In a graph satisfying $\mathscr{E}_2$, a trail can be extended to a two-sided infinite Eulerian trail if and only if it is bi-extensible.

Our proof of Theorem A is self-contained, and yields a relatively short proof of the Erdős, Grünwald and Weiszfeld theorem (Corollary 4.6 and Corollary 4.12).

As we explained before, we apply Theorem A to prove some computability results in relation to the Erdős, Grünwald and Weiszfeld theorem. These results are valid for the class of graphs satisfying the following condition.

Definition. A **moderately computable graph** is a computable graph for which the vertex degree function $V(G) \to \mathbb{N} \cup \{\infty\}$ is computable.

For us, a computable graph is one whose vertex and edge sets are indexed by decidable subsets of $\mathbb{N}$ in such a manner that adjacency and incidence relation are decidable on these indices (Definition 5.1). The condition on the incidence relation is needed as we allow our graphs to have loops and multiple edges. The class of moderately computable graphs generalizes that of highly computable graphs in the sense that they coincide for locally finite graphs.

In contrast to the hypothesis of highly computable graph, the hypothesis of moderately computable graph does not ensure the computability of the distance function on the vertex set. Despite of this, we shall prove that the hypothesis of moderately computable graph is sufficient to decide whether a finite trail is bi-extensible or right-extensible.

Theorem B. *In a moderately computable graph satisfying $\mathscr{E}_1$ (resp. $\mathscr{E}_2$), it is algorithmically decidable whether a finite trail can be extended to a one-sided (resp. two-sided) infinite Eulerian trail.*

Now we make some coments on the uniformity of the algorithms in Theorem B. Some of our algorithms are uniform, while others require some finite information about the graph which can not be computed from its description.

Our algorithm for two-sided infinite Eulerian trails is always uniform. However, our algorithm for one-sided infinite Eulerian trails is uniform only for highly computable graphs. The reason being that a moderately computable graph satisfying $\mathscr{E}_1$ may contain or may not a vertex with odd degree, and this information is needed by the algorithm. This result can not be improved, in the sense that it is not possible to decide from the description of a moderately computable graph satisfying $\mathscr{E}_1$, whether it contains a vertex with odd degree or not.

We now turn to the sets $\mathscr{E}_1(G)$ and $\mathscr{E}_2(G)$ of one-sided and two-sided infinite Eulerian trails of G, defined as subsets of $\mathbb{N}^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{Z}}$, respectively. Our results for trails are naturally translated or expressed as a property of these sets.

Theorem C. *In a moderately computable graph G which satisfies $\mathscr{E}_1$ (resp. $\mathscr{E}_2$), the set $\mathscr{E}_1(G)$ (resp. $\mathscr{E}_2(G)$) is effectively closed and co-effectively closed.*

A closed set X contained of $\mathbb{N}^{\mathbb{N}}$ or $\mathbb{N}^{\mathbb{Z}}$ is called effectively closed (co-effectively closed) when there is an algorithm which on input a cylinder set, halts if and only if it intersects (does not intersect) X. These are computability properties for sets in the context of computable analysis in metric spaces, and in a sense it is the strongest among different computability notions for closed sets [3]. It is a general fact that a set with this property has a dense subset of computable points [10, Proposition 2.3.2]. It follows that Theorem C implies Bean's result, which in this language asserts that if a highly computable graph G satisfies $\mathscr{E}_1$ (resp. $\mathscr{E}_2$), then the set $\mathscr{E}_1(G)$ (resp. $\mathscr{E}_2(G)$) has at least one computable point.

2.1. **Paper structure.** In Section 3 we review some basic facts and terminology from graph theory. In Section 4 we prove Theorem A. In Section 5 we review some computability notions for infinite graphs, and prove Theorem B and Theorem C.

2.2. **Acknowledgements.** The author is grateful to Rod Downey for pointing out the results in Bean's paper. The author is also grateful to Vittorio Cipriani, Vallentino Delle Rose, and Cristóbal Rojas for fruitful discussions which derived in significant improvements from a first version of this article, in particular concerning Proposition 4.11 and the notion of moderately computable graph. The author is also grateful to Sebastián Barbieri and Paola Rivera for their helpful comments.

This research was partially supported by ANID CONICYT-PFCHA/Doctorado Nacional/2020-21201185, ANID/Basal National Center for Artificial Intelligence CENIA FB210017, and the European Union's Horizon 2020 research and innovation program under the Marie Sklodowska-Curie grant agreement No 731143.

3. Preliminaries

3.1. **Graph theory.** Throughout this paper we deal with finite and infinite undirected graphs, where two vertices may be joined by multiple edges and self loops are allowed. The vertex set of a graph G is denoted by $V(G)$, and its edge set by $E(G)$.

Each edge **joins** a pair of vertices, and is said to be **incident** to these vertices. Two vertices joined by an edge are called **adjacent**. A **loop** is an edge joining a vertex to itself. The **degree** $\deg_G(v)$ of the vertex v in the graph G is the number of edges incident to v, where loops are counted twice.

It will be convenient for us to define trails as graph homomorphisms. **A graph homomorphism** $f\colon G' \to G$ is a function which sends vertices to vertices, edges to edges, and is compatible with the incidence relation.

We denote by $[\![a,b]\!]$ the graph with vertex set $\{a,\dots,b\} \subset \mathbb{Z}$, and with edges $\{c,c+1\}$ joining c to $c+1$, for $c \in \{a,\dots,b-1\}$. The graphs $[\![\mathbb{N}]\!]$ and $[\![\mathbb{Z}]\!]$ are defined in a similar manner. We assume that 0 belongs to $\mathbb{N}$.

A **trail** (resp. **one-sided infinite trail**, **two-sided infinite trail**) on G is a graph homomorphism $t\colon [\![a,b]\!] \to G$ (resp. $t\colon [\![\mathbb{N}]\!] \to G$, $t\colon [\![\mathbb{Z}]\!] \to G$) which does not repeat edges. In these cases we say that t **visits** the vertices and edges in its image, and we call it **Eulerian** when it visits every edge of G exactly once.

Let $t\colon [\![a,b]\!] \to G$ be a trail. We say that $t(a)$ is its **initial vertex** and $t(b)$ is its **final vertex**. We say that t joins $t(a)$ to $t(b)$, and we call it **closed** when $t(a) = t(b)$.

In several constructions we will consider subgraphs induced by sets of edges. The subgraph **induced** by a set of edges $E \subset E(G)$ is denoted $G[E]$. Its edge set is E and its vertex set is the set of all vertices incident some edge in E. For a set of edges $E \subset E(G)$, $G - E$ denotes the subgraph induced by the set of edges $E(G) - E$. Given a trail t we denote by $E(t)$ the set of edges visited by t, and we denote by $G - t$ the induced subgraph $G - E(t)$. We say that $G - t$ is obtained by **removing** t from G. We denote by $\deg_t(v)$ the degree of v in the graph induced by $E(t)$.

A graph is said to be **finite** if its edge set is finite, **even** if every vertex has finite and even degree, **locally finite** if every vertex has finite degree, and **connected** if any two vertices are joined by a trail. The vertex set of a connected graph is a metric space with the trail-length distance, denoted d_G. A **connected component** in G is a connected subgraph of G which is maximal for the subgraph relation. As defined in the introduction, the **number of ends** of a graph G is the supremum of infinite connected components of $G - E$, where E ranges over all finite subsets of $E(G)$.

We recall that **Euler's theorem** asserts that a finite and connected graph G admits a closed Eulerian trail if and only if all vertices have even degree, and an Eulerian trail from a vertex u to $v \neq u$ if and only if these are the only vertices with odd degree. The **Handshaking lemma** asserts that the sum of all vertex degrees of a finite graph G equals twice the cardinality of $E(G)$, and in particular is an even number.

3.2. **Concatenation and inversion of trails.** We introduce some ad hoc terminology regarding "pasting" trails. Let $t\colon [\![a, b]\!] \to G$ and $s\colon [\![c, d]\!] \to G$ be edge disjoint trails. If the final vertex of t is also the initial vertex of s, the **concatenation of** s **at the right** of t is the trail whose domain is $[\![a, b + d - c]\!]$, whose restriction of r to $[\![a, b]\!]$ equals t, and the restriction of r to $[\![b, b + d - c]\!]$ follows the same path as s, but with the domain shifted. If the final vertex of s coincides with the initial vertex of t, we define the **concatenation of** s **at the left of** t as the trail whose domain is $[\![a - (d - c), b]\!]$, which on $[\![a - (d - c), a]\!]$ follows the path of s but with the domain shifted, and on $[\![a, b]\!]$ follows the path of t.

We say that a trail **extends** t if its restriction to the domain of t is equal to t. For example, if we concatenate a trail at the right or left of t, we obtain a trail which extends t. Finally we define the **inverse** of t, denoted $-t$, as the trail with domain $[\![-b, -a]\!]$ and which visits the vertices and edges visited by t in but in inverse order.

4. Proof of Theorem A

In this section we prove Theorem A. We start with a technical Lemma which will be used for graphs satisfying both $\mathscr{E}_1$ and $\mathscr{E}_2$.

Lemma 4.1. *Let G be a connected graph, and let t be a trail on G such that every vertex different from the initial or final vertex of t has even or infinite degree in G. Then there is a trail which visits all vertices and edges visited by t, with the same initial and final vertices as t, and whose remotion from G leaves no finite connected component.*

Proof. Let G' be the subgraph of G induced by the edges visited by t and the edges in finite connected components of $G - t$. Observe that G' is finite as there is at most one finite connected components in $G - t$ for each vertex visited by t. It is clear that G' is connected, and that $G - G'$ has no finite connected component.

Let u be the initial vertex of t, and let v be the final vertex of t. We show that all vertices in G' different from u or v have even degree in G'. Then we show that u and v have both even degree in G' if they are equal, and otherwise both have odd degree in G'. This proves the claim by Euler's theorem.

Let w be a vertex in G' not visited by t, so $\deg_G(w) = \deg_{G'}(w)$. As G' is a finite graph it follows that $\deg_G(w)$ is finite, and then the hypothesis on G implies that $\deg_{G'}(w)$ is an even number. Now let w be a vertex visited by t, but different from u and v. We verify that $\deg_{G'}(w)$ is finite and even. Indeed, there is at most one connected component of $G - t$ containing w. If this connected component is infinite, then there is no finite connected component of $G - t$ containing w, and thus $\deg_{G'}(w) = \deg_t(w)$, a finite and even number. If this connected component is finite it follows that $\deg_G(v)$ is finite, and then even by hypothesis. It follows that $\deg_{G'}(v)$ is also even. We now adress u and v. If our claim on the degrees of u and

v on G' fails then G' would have exactly one vertex with odd degree, contradicting the handshaking lemma. □

4.1. **The case of one-sided infinite trails.** In this subsection we prove Theorem A for graphs satisfying $\mathscr{E}_1$, that is, that a trail can be extended to a one-sided infinite Eulerian trail if and only if it is right-extensible. The structure of the proof is very simple, we show that right-extensible trails exist, and that they can be extended to larger right-extensible trails. This allows an iterative construction. A key idea is that the remotion of a right-extensible trail from a graph satisfying $\mathscr{E}_1$ leaves a graph satisfying $\mathscr{E}_1$.

Lemma 4.2. *Let G be a graph satisfying $\mathscr{E}_1$. If t is a right-extensible trail in G then $G - t$ also satisfies $\mathscr{E}_1$. Moreover, the final vertex of t is distinguished in $G - t$.*

Proof. The proof of this result is a case by case review of the vertex degree of the initial and final vertex of t in both G and $G - t$. □

Lemma 4.3. *Let G be a graph satisfying $\mathscr{E}_1$, and let v be a distinguished vertex in G. Then for any edge e there is a right-extensible trail on G whose initial vertex is v and which visits e.*

Proof. As G is connected there is a trail $t\colon [\![0, c]\!] \to G$ with $t(0) = v$ and which visits e. By Lemma 4.1 we can asume that the remotion of t leaves no finite connected component, and thus $G - t$ is a connected graph.

We claim that t is right-extensible. The first and second conditions in the definition hold by our choice of t. For the third condition we separate the cases where t is closed or not. If t is a closed trail then $\deg_t(t(b))$ is even while $\deg_G(t(b))$ is either odd or infinite, as $t(0) = t(b)$ is distinguished. If t is not a closed trail then $\deg_t(t(b))$ is odd while $\deg_G(t(b))$ is either even or infinite. In both cases it follows that $t(b)$ has edges in G not visited by t, so the third condition in the definition of right-extensible is also satisfied. □

Lemma 4.4. *Let G be a graph satisfying $\mathscr{E}_1$. Then for any right-extensible trail t and edge e there is a trail which is right-extensible on G, extends t, and visits e.*

Proof. By Lemma 4.2 the graph $G-t$ satisfies $\mathscr{E}_1$ and contains the final vertex of t as a distinguished vertex. Now by Lemma 4.3 the graph $G-t$ admits a right-extensible trail $s : [\![0, c]\!] \to G - t$ which starts at the final vertex of t, and which visits e. Thus the trail $t' : [\![0, b + c]\!] \to G$ obtained by concatenating s at the right of t is right-extensible in G, visits e, and extends t. □

The following result proves Theorem A for graphs satisfying $\mathscr{E}_1$.

Proposition 4.5. *Let G be a graph satisfying $\mathscr{E}_1$. Then a trail on G is right-extensible if and only if it can be extended to a one-sided infinite Eulerian trail. Moreover, a vertex is distinguished in G if and only if it is the initial vertex of a one-sided infinite Eulerian trail on G.*

Proof. Let t be a right-extensible trail, and let $(e_n)_{n\in\mathbb{N}}$ be a numbering of $E(G)$. In order to extend t to a one-sided infinite Eulerian trail we just iterate Lemma 4.4 to obtain a sequence of trails $(t_n)_{n\in\mathbb{N}}$ such that t_0 extends t, for each n the trail t_n visits e_n, and for each n the trail t_{n+1} extends t_n. This sequence defines a one-sided infinite Eulerian trail.

Let v be a distinguished vertex. Then Lemma 4.3 shows that it is the initial vertex of a right-extensible trail, and then the argument above shows that v is the initial vertex of a one-sided infinite Eulerian trail.

For the remaining implications, let T be a one-sided infinite Eulerian trail on G. We claim that $T(0)$ is distinguished. Indeed, it is clear that $T(0)$ can not have finite even degree, and that if $T(0)$ has finite degree then every other vertex has even or infinite degree. Moreover, it is clear from the definition that the restriction of T to a domain of the form $[\![0, n]\!]$ is a right-extensible trail. $\square$

Corollary 4.6. *A graph admits a one-sided infinite Eulerian trail if and only if it satisfies $\mathscr{E}_1$.*

Proof. If a graph satisfies $\mathscr{E}_1$, then it admits a one-sided infinite Eulerian trail by Proposition 4.5.

Now let G be a graph which admits a one-sided infinite Eulerian trail T. It is clear then that G satisfies the first and third conditions of $\mathscr{E}_1$. A counting argument considering the number of times that T goes "in" and "out" a vertex shows that the initial vertex of T must have either odd or infinite degree, and every other vertex must have either even or infinite degree. $\square$

4.2. **The case of two-sided infinite trails.** In this subsection we prove Theorem A for graphs satisfying $\mathscr{E}_2$, that is, that a trail can be extended to a two-sided infinite Eulerian trail if and only if it is bi-extensible. The structure of the proof is very simple, we show that bi-extensible trails exist, and can be extended to larger bi-extensible trails. A key idea is that the remotion of a bi-extensible *closed* trail from a graph satisfying $\mathscr{E}_2$ leaves a graph satisfying $\mathscr{E}_2$.

Lemma 4.7. *Let G be a graph satisfying $\mathscr{E}_2$ and let t be a bi-extensible closed trail. Then $G - t$ also satisfies $\mathscr{E}_2$.*

Proof. Note that $G - t$ is connected. Indeed as t is bi-extensible, $G - t$ has no finite connected components, and by the third condition in $\mathscr{E}_2$ the graph $G - t$ has at most one infinite connected component. Joining these two facts, we conclude that $G - t$ is connected. The remaining conditions in $\mathscr{E}_2$ are easily verified. $\square$

Lemma 4.8. *Let G be a graph satisfying $\mathscr{E}_2$. Then for any vertex v and edge e there is a bi-extensible trail which visits v and e.*

Proof. By connectedness of G there is a trail $t : [\![0, b]\!] \to G$ which visits both v and e. By Lemma 4.1 we can assume that the remotion of this trail leaves no finite connected component in G. We now consider two cases.

In the first case t is not closed. We assert that then it is bi-extensible. Indeed, as $t(0)$ and $t(b)$ have odd degree in t, there are edges e and f not visited by t, and incident to $t(0)$ and $t(b)$ respectively. We can take $e \neq f$ because otherwise the graph $G - t$ would be forced to have a finite connected component, contradicting our choice of t.

In the second case t is a closed trail. We just reparametrize this trail in order to obtain a bi-extensible trail. As G is connected, t visits a vertex u which lies in $G - t$. As the degree of u in $G - t$ is even, there are at least two different edges in $G - t$ which are incident to u. This holds even if some edge incident to u in $G - t$ is a loop. We simply reparametrize t so that it becomes a closed trail with u as initial and final vertex. It is clear that this new trail is bi-extensible. $\square$

Lemma 4.9. *Let G be a graph satisfying $\mathscr{E}_2$. Then for any bi-extensible trail t and edge e there is a bi-extensible trail which extends t and visits e. We can choose this extension of t so that its domain strictly extends the domain of t in both directions.*

Proof. Let t and e be as in the statement. Clearly it suffices to prove the existence of a bi-extensible trail s which extends t, visits e, and whose domain strictly extends the domain of t in a direction of our choice. In order to prove our claim we consider three cases.

In the first case $t\colon [\![a,b]\!] \to G$ is a closed trail. Then $G - t$ is connected by the third condition in $\mathscr{E}_2$ and the graph $G - t$ satisfies $\mathscr{E}_2$ by Lemma 4.7. We apply Lemma 4.8 to the graph $G - t$ to obtain a trail $t_1\colon [\![a_1,b_1]\!] \to G - t$ which visits the vertex $t(a) = t(b)$, the edge e, and is bi-extensible on $G - t$. We just need to split t_1 in two trails and concatenate them to t. For this let $c_1 \in [a_1, b_1]$ be such that $t_1(c_1) = t(a)$, and define l_1 and r_1 as the restrictions of t_1 to $[\![a_1, c_1]\!]$ and $[\![c_1, b_1]\!]$, respectively. We define the trail s by concatenating l_1 to the left of t, and then r_1 to its right. By our choice of t_1 and c_1 it follows that s visits e, extends t, and it is bi-extensible on G. An alternative way to define the trail s is by concatenating $-r_1$ at the left of t, and $-l_1$ to its right.

Observe that it is possible that c_1 equals a_1 or b_1, in this situation the domain of s extends that of t only in one direction. We can choose this direction with the two possible definitions of s.

In the second case $t\colon [\![a,b]\!] \to G$ is not closed while $G - t$ is connected. We show that we can extend t to a bi-extensible closed trail s, and then we go back to the first case. As $G - t$ is connected we can take a trail $t_2\colon [\![a_2,b_2]\!] \to G - t$ whose initial vertex is $t(a)$ and whose final vertex is $t(b)$. By Lemma 4.1 we can assume that $(G - t) - t_2$ has no finite connected components. We just need to split t_2 in two trails and concatenate them to t as follows. Let $c_2 \in [a_2, b_2]$ be such that $t_2(c_2)$ lies in $(G - t) - t_2$, that is, $t_2(c_2)$ is a vertex with incident edges not visited by t_2. Observe that there must be at least two such edges by the parity of the vertex degrees. We define l_2 and r_2 as the restrictions of t_2 to $[\![a_2, c_2]\!]$ and $[\![c_2, b_2]\!]$, respectively. Note that the final vertex of $-l_2$ is $t(a)$, and the initial vertex of $-r_2$ is $t(b)$. We define s by concatenating $-l_2$ to the left of t, and then $-r_2$ to its right. By our choice of t_2 and c_2, s is a bi-extensible closed trail which extends t. This concludes our proof of the second case.

In the third case, t is not closed and $G - t$ is not connected. It follows that $G - t$ has two infinite connected components and no finite connected component. These components satisfy $\mathscr{E}_1$, and have the initial and final vertex of t as a distinguished vertex. We simply apply Lemma 4.3 on each one of these components and then concatenate to extend t as desired. $\square$

The following result proves Theorem A for graphs satisfying $\mathscr{E}_2$.

Proposition 4.10. *Let G be a graph satisfying $\mathscr{E}_2$. Then a trail is bi-extensible if and only if it can be extended to a two-sided infinite Eulerian trail.*

Proof. Let t be a bi-extensible trail, and let $(e_n)_{n\in\mathbb{N}}$ be a numbering of the edges in $E(G)$. We iterate Lemma 4.9 in order to obtain a sequence of bi-extensible trails $(t_n)_{n\in\mathbb{N}}$ such that for each n the trail t_n visits e_n and t_{n+1} extends t_n. This sequence defines a two-sided infinite Eulerian trail on G which extends t.

For the other direction, it is clear that the restriction of a two-sided infinite Eulerian trail $T : [\![\mathbb{Z}]\!] \to G$ to a set of the form $[\![n, m]\!]$ yields a bi-extensible trail. $\square$

We now prove an alternative characterization of which trails can be extended to two-sided infinite Eulerian trails. This will be relevant in Section 5 to prove a decidability result.

Proposition 4.11. *Let G be a graph satisfying $\mathscr{E}_2$. Then a trail t can be extended to a two-sided infinite Eulerian trail on G if and only if*

(1) Every connected component of $G - t$ contains either the initial or final vertex of t.
(2) There is an edge e incident to the final vertex of t which was not visited by t.
(3) There is an edge $f \neq e$ incident to the initial vertex of t which was not visited by t.

Proof. By Proposition 4.10 it suffices to prove that a trail satisfies these conditions if and only if it is bi-extensible.

Let t be a trail satisfying these conditions, we claim that $G - t$ has no finite connected components and thus t is bi-extensible. This is clear if t is a closed trail. If t is not closed, we let G_- (resp. G_+) be the connected component of $G - t$ containing the initial (resp. final) vertex of t. If G_+ is equal to G_-, it follows that $G - t$ has no finite connected component. If G_+ is different to G_-, we claim claim that both must be infinite, and thus $G - t$ has no finite connected component. Observe that as G satisfies $\mathscr{E}_2$, the initial and final vertex of t are the only vertices in $G - t$ with odd degree. Now if G_+ (or G_-) is finite, then it has exactly one vertex with odd degree, a contradiction by handshaking lemma. Our claim that t is bi-extensible follows.

Now let t be a bi-extensible trail, we claim that it satisfies these conditions. The fact that t can be extended to a bi-infinite Eulerian trail (4.10) implies that every vertex in $G - t$ can be joined by a trail to the initial or the final vertex of t, so it follows that every connected component of $G - t$ contains either the initial or final vertex of t. □

Corollary 4.12. *A graph admits a two-sided infinite Eulerian trail if and only if it satisfies $\mathscr{E}_2$.*

Proof. If a graph satisfies $\mathscr{E}_2$, then it admits a two-sided infinite Eulerian trail by Lemma 4.8 and Proposition 4.10.

Now let G be a graph which admits a two-sided infinite Eulerian trail T. It is clear that G satisfies the first and second condition in $\mathscr{E}_2$. We verify the third condition in $\mathscr{E}_2$.

Let E be a finite set of edges which induces an even subgraph of G, we claim that $G - E$ has one infinite connected component. For this let u and v be the first and last vertex in $G[E]$ visited by T. If $u = v$ then our claim follows. If $u \neq v$ then we take F to be the set of edges visited by T after u but before v, so $E \subset F$. Observe that $G[F] - E$ is a graph where u and v have odd degree, and the remaining vertices have even degree. If $G[F] - E$ is connected then our claim follows. This is the only possibility: if $G[F] - E$ is not connected then a connected component of $G[F] - E$ containing u is a finite and connected graph containing exactly one vertex with odd degree, a contradiction by handshaking lemma. □

5. Computability results

In this section we prove Theorem B and Theorem C. We assume some familiarity with basic concepts in computability or recursion theory, such as decidable subset of $\mathbb{N}$ and computable function on natural numbers. By algorithm we refer to the formal object of Turing machine. In some cases we consider algorithms treating with finite objects different than natural numbers, such as finite graphs or finite sets of vertices. These objects will be assumed to be described by natural numbers in a canonical way. A more formal treatment of these computations can be done through the theory of numberings, but we avoid this level of detail here.

We start with a review of standard computability notions for infinite graphs, and introducing the notion of moderately computable graph.

5.1. Computable, moderately computable, and highly computable graphs.

Let us recall that the adjacency relation of a graph G is the set

$$\{(u, v) \in V(G)^2 \mid u \text{ and } v \text{ are adjacent}\}.$$

A common approach in the literature is to call a graph computable when its vertex set is a decidable subset of $\mathbb{N}$ and its adjacency relation is a decidable subset of $\mathbb{N}^2$. This approach only works for simple graphs, that is, graphs with no loops or multiple edges between vertices. In order to omit this restriction we also consider the incidence relation between edges and pairs of vertices

$$\{(e, u, v) \in E(G) \times V(G)^2 \mid e \text{ joins } u \text{ and } v\}.$$

Definition 5.1. A **computable** graph is a graph G whose edge and vertex sets are endowed with an indexing or numbering by decidable sets of natural numbers I and J, $E(G) = \{e_i \mid i \in I\}$, $V(G) = \{v_j \mid j \in J\}$, such that the relations of adjacency and incidence between edges and pairs of vertices are decidable on these indices.

The fact that a locally finite graph is computable does not imply that the vertex degree function is computable [4]. This is the reason behind the following definition, which is standard in the literature of recursive combinatorics.

Definition 5.2. A **highly computable** graph is a computable graph which is locally finite, and for which the vertex degree function $\deg_G \colon V(G) \to \mathbb{N}$ is computable.

Here we consider following class of graphs, which can be considered as a natural generalization of highly computable graphs to graphs with vertex of infinite degree.

Definition 5.3. A **moderately computable** graph is a computable graph for which the vertex degree function $\deg_G \colon V(G) \to \mathbb{N} \cup \{\infty\}$ is a computable function.

We will also make use of the following notion. A **description** of a computable graph G where $E(G) = \{e_i \mid i \in I\}$ and $V(G) = \{v_j \mid j \in J\}$, is a tuple of algorithms deciding membership in I, J, and the relation of incidence between pairs of vertices and edges. A description of a highly computable graph and moderately computable graph contains, additionally, an algorithm for the computable function $n \mapsto \deg_G(v)$.

5.2. **Some results regarding moderately computable graphs.** In this subsection we prove the computability of some operations on moderately computable graphs. We start by reviewing basic procedures on the finite subgraphs of a moderately computable graph G.

Given a vertex v, a distance $r \in \mathbb{N}$, and a precision $s \in \mathbb{N}$ we define $G(v, r, s)$ as the finite and connected subgraph of G induced by the edges of all trails in G which visit v, have length at most r, and which only visit edges among $\{e_1, \dots, e_s\}$.

The graph $G(v, r, s)$ can be computed from v, r and s. We just need to compute the finite set of all possible trails with edges $\{e_1, \dots, e_s\}$, and check which of these trails satisfy the conditions. In this manner we obtain the edge set of $G(v, r, s)$, which then determines its vertex set.

Let $G(v, r)$ be the subgraph of G induced by all trails in G which visit v and have length at most r. The graph $G(v, r)$ is approximated by the graphs $G(v, r, s)$ in the following sense:

$$G(v, r) = \bigcup_{s \in \mathbb{N}} G(v, r, s).$$

The hypothesis of moderately computable graph is sufficient to decide whether the graph $G(v, r)$ is finite. In order to do this we first check whether $\deg_G(v)$ is finite. If it is finite, then we compute the finite set of all vertices which are adjacent to v. Then we repeat the process for the vertices obtained, and so on. If we find some vertex of infinite degree in less than r steps, then we conclude that $G(v, r)$ is infinite, and otherwise we conclude that $G(v, r)$ is finite.

The hypothesis of moderately computable graph is sufficient to decide, given a finite graph $G(v, r, s)$, whether a vertex u in this graph has all its incident edges in G appearing in $G(v, r, s)$. For this we just need to compare $\deg_G(u)$ with $\deg_{G(v,r,s)}(u)$. In this manner we can check whether $G(v, r)$ is equal to $G(v, r, s)$ for some s.

Joining these two procedures we have an algorithm which on input v, r decides whether $G(v, r)$ is finite, and if it is a finite graph, outputs a the graph $G(v, r)$. In this manner we recover the standard fact that in a highly computable graph G, we can compute the finite graph $G(v, r)$ from v and r.

We now turn to prove some facts related to the remotion of finite sets of edges from a graph. This is related to the concept of ends, and to the properties for trails that we defined in the introduction. More specifically, it is known that on a highly computable graph with one end, it is algorithmically decidable whether the remotion of a finite set of edges leaves a connected graph [5, Lemma 4.4]. We will prove that this result also holds on moderately computable graphs.

Lemma 5.4. *There is an algorithm which on input the description of a moderately computable graph G and a finite set of edges E in G, outputs a list of all vertices in $G - E$ which are incident to some edge in E.*

Proof. On input G and E we can compute the finite set V of all vertices in $V(G)$ which are incident to some edge in E. Then for each $v \in V$, we can decide whether v lies in $G - E$ as follows. If $\deg_G(v) = \infty$, then v lies in $G - E$ because E is finite. Otherwise $\deg_G(v)$ is a finite number and we can use it to compute the set of all edges in $E(G)$ incident to v. Then we just check whether all these edges have been removed in $G - E$. □

Lemma 5.5. *There is an algorithm which on input the description of a moderately computable graph G and a finite set of edges E in G, halts if and only if $G - E$ has a finite connected component.*

Proof. On input G and E, the algorithm proceeds as follows. First, compute a vertex v which is incident to some edge in E. For each pair of natural numbers r and s, we compute the graph $G(v, r, s)$ and check whether the finite graph $G(v, r, s) - E$ satisfies the following.

- $G(v, r, s) - E$ has a conneted component G' verifying the following two conditions.
- Every vertex in G' has finite degree in G.
- Every vertex in G' has all its incident edges from $E(G)$ in $E(G(v, r, s))$.

If such a connected component exists, then the algorithm halts and concludes that $G - E$ has a finite connected component.

Let us check that this algorithm is correct. If the algorithm halts and such a component G' exists for some r_0 and s_0, then G' will continue to be a finite connected component of $G(v, r, s) - E$ for all $r \geq r_0$, $s \geq s_0$ because no new edges incident to vertices in G' can appear, thus G' is a finite connected component of $G - E$. On the other hand it is clear that if $G - E$ has some connected component G', then it has the form specified above for r and s big enough. □

Lemma 5.6. *There is an algorithm which on input the description of a moderately computable graph with one end G and a finite set of edges E in G, decides whether $G - E$ is connected.*

Proof. By Lemma 5.5, it suffices to exhibit an algorithm which on input a description of G and E as in the statement, halts if and only if $G - E$ is connected.

On input G and E, the algorithm proceeds as follows. First we use the procedure in to compute the finite set $\{v_1, \ldots, v_n\}$ of all vertices incidient to some edge in E, and which also lie in $G - E$. Then we define G_i as the connected component in $G - E$ containing v_i, $i \in \{1, \ldots, n\}$.

Observe that the graph $G - E$ is connected if and only if all the components G_i are equal. Moreover, a component G_i equals G_j if and only if there is a trail in

G whose initial vertex is v_i, final vertex v_j, and which does not visit edges among E. We just need to search for trails exhaustively, and halt the algorithm when, for every pair of $i \neq j \in \{1, \ldots, n\}$, we have found a trail whose initial vertex is v_i, whose initial vertex is v_j, and which does not visit edges in E. □

5.3. **Decidability of the properties bi-extensible and right-extensible.** In this subsection we prove Theorem B, which asserts that for each moderately computable graph G satisfying $\mathscr{E}_1$ (resp. $\mathscr{E}_2$) it is algorithmically decidable whether a finite trail can be extended to a one-sided (resp. two-sided) infinite Eulerian trail on G.

Some of our algorithms are uniform on the graph, while others require some finite information about the graph which can not be obtained computably from its description. More precisely, the algorithm for two-sided infinite Eulerian trails is always uniform, while the algorithm for one-sided infinite Eulerian trails is uniform only for highly computable graphs. The reason is that for moderately computable graphs, the algorithm requires the information of whether the graph has a vertex with odd degree or not.

Proposition 5.7. *There is an algorithm which on input the description of a moderately computable graph G satisfying $\mathscr{E}_1$, the information of whether G has some vertex with odd degree, and a finite trail, decides whether the trail can be extended to a one-sided infinite trail on G.*

Proof. By Proposition 4.5, it suffices to verify whether t is right-extensible. We sketch an algorithm which on input G and t, checks the three conditions in Definition 2.5.

The first condition asserts that $G - t$ is connected. We check this with the procedure in Lemma 5.6.

The second condition asserts that the initial vertex v of t is distinguished (Definition 2.4). We use the computability of $\deg_G$ to verify whether this condition holds. If the algorithm is informed that the graph has a vertex with odd degree, then we just need to check whether $\deg_G(v)$ is odd. If the algorithm is informed that G has no vertex of odd degree, then we just check whether $\deg_G(v)$ is infinite.

The third condition asserts that the final vertex of t has some edge not visited by t. This can be checked with the procedure in Lemma 5.4. □

Proposition 5.8. *There is an algorithm which on input the description of a highly computable graph G satisfying $\mathscr{E}_1$ and a finite trail, decides whether it is right-extensible.*

Proof. Let us recall that a locally finite graph satisfying $\mathscr{E}_1$ has exactly one vertex with odd degree (Definition 2.2). On input the graph G and a trail t, we just have to use the algorithm in Proposition 5.7, and inform the algorithm that the input graph G has a vertex with odd degre. □

Proposition 5.9. *There is an algorithm which on input the description of a moderately computable graph G satisfying $\mathscr{E}_2$ and a finite trail, decides whether it can be extended to a two-sided infinite Eulerian trail.*

Proof. By Proposition 4.10, it suffices to check whether the input trail is bi-extensible on the graph. We sketch an algorithm which on input G and t, verifies the three conditions in Definition 2.6.

We start with the second and third conditions in Definition 2.6, that is, we check whether there are two edges $e \neq f$ in G which are not visited by t, and are incident to the initial and final vertices of t, respectively. Let u and v be these vertices. We just need to use the algorithm in Lemma 5.4, and check whether u and v lie in the output of this algorithm. After this is done, we just review how many edges incident to u and v are left in $G - E$.

Once we checked the second and third conditions in Definition 2.6, it only remains to check the first condition to decide whether t is bi-extensible or not. Let us recall this condition, which we state now as i).

i) $G - t$ contains no finite connected component.

We will use Proposition 4.11, which asserts that this is equivalent to the following condition.

ii) Every connected component of $G - t$ contains either the initial or the final vertex of t.

In order to check whether these these two equivalent conditions are satisfied by t, we run two procedures simultaneously. The first halts when t is not bi-extensible, and the second halts when t is bi-extensible.

The first procedure is the algorithm in Lemma 5.5 with input G and $E(t)$. This algorithm halts if and only if $G - t$ contains some finite connected component, that is, if t fails to satisfy i). The second procedure is new, but is very similar to the algorithm in Lemma 5.4. Indeed, we only need to list all trails in G which also lie in $G - t$, and stop the procedure once we have found that ii) is satisfied. We describe this procedure in more detail for completeness.

First we use the algorithm in Lemma 5.4 to compute the finite set $\{v_1, \dots, v_n\}$ of all vertices incident to some edge in $E(t)$ which also lie in $G - t$, and define G_i as the connected component in $G - t$ containing v_i. We label these vertices so that v_1 is the initial vertex of t, and v_n is the final vertex of t. Then we list all trails in G, and stop the procedure once we have found that for every $i \in \{2, \dots, n-1\}$ there is a trail which do not visit edges from $E(t)$, and joins v_i to either v_1 or v_n, □

5.4. **The sets of infinite Eulerian trails.** In this subsection we prove Theorem C. For this we define the sets of one-sided and two-sided infinite Eulerian trails of a graph G, denoted $\mathscr{E}_1(G)$ and $\mathscr{E}_2(G)$, as closed subsets of $\mathbb{N}^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{Z}}$. Then we prove that these sets have a strong computability property using our decidability results for trails. We start with the set $\mathscr{E}_1(G)$ of one-sided infinite Eulerian trails.

Definition 5.10. Let G be a computable graph. We define $\mathscr{E}_1(G)$ as the set of all functions $x\colon \mathbb{N} \to \mathbb{N}$ satisfying the following properties.

- For each even number n, G has a vertex $v_{x(n)}$ with index $x(n)$, and for each odd number m, G has an edge $e_{x(m)}$ with index $x(m)$.
- For each odd number n, the edge $e_{x(n+1)}$ joins $v_{x(n)}$ to $v_{x(n+2)}$.
- The graph homomorphism $T\colon [\![\mathbb{N}]\!] \to G$ defined by $T(n) = v_{x(2n)}$, $T\{n, n+1\} = e_{x(2n+1)}$ is a one-sided infinite Eulerian trail on G.

The idea behind the definition is that in a sequence x, each odd number n is interpreted as the edge whose index is $x(n)$ and each even number m is interpreted as the vertex with index $x(m)$. This defines a sequence of edges and vertices, which is required to be compatible with the incidence relation, and is also required to be an infinite Eulerian trail.

Of course the set $\mathscr{E}_1(G)$ is nonempty if and only if G satisfies $\mathscr{E}_1$. We now proceed with the proof of Theorem C in the case $\mathscr{E}_1$.

Proposition 5.11. *Let G be a moderately computable graph satisfying $\mathscr{E}_1$. There is an algorithm which on input a finite sequence of natural numbers $(a_i)_{0\leq i\leq n}$, decides whether the cylinder set*

$$\{x \in \mathbb{N}^{\mathbb{N}} \mid x(0) = a_0, \dots, x(n) = a_n\}$$

intersects $\mathscr{E}_1(G)$.

Proof. We start by describing the procedure on input $(a_i)_{0\leq i\leq n}$ with n even. Using the decidability of the incidence relation, we check whether the sequence of edges and vertices

$$v_{a_0}e_{a_1}\dots e_{a_{n-1}}v_{a_n}$$

describes a trail, i.e. a graph homeomorphism $t\colon [\![0,\frac{n}{2}]\!]\to G$ which does not repeat edges. If this sequence does not describe a trail, then we conclude that the corresponding cylinder set does not intersect $\mathscr{E}_1(G)$. If this sequence indeed defines a graph homeomorphism, then we proceed to check whether t is a right-extensible trail with the algorithm from Proposition 5.8, and the result of this algorithm determines whether the corresponding cylinder set intersects $\mathscr{E}_1(G)$.

We now describe the procedure on input a sequence $(a_i)_{0\leq i\leq n}$ with n odd, that is, a_n represents an edge. In this case we can use the decidability of the incidence relation to compute a natural number a_{n+1} satisfying the following condition: a_{n+1} is the index of a vertex $v_{a_{n+1}}$ such that e_{a_n} joins $v_{a_{n-1}}$ and $v_{a_{n+1}}$. Observe that this condition only depends on a_{n-1} and a_n. Now we have a finite sequence $(a_i)_{0\leq i\leq n+1}$ with $n+1$ even, and we proceed as described before. ☐

We now consider the set of two-sided infinite Eulerian trails $\mathscr{E}_2(G)$ of a computable graph G, which is a subset of $\mathbb{N}^{\mathbb{Z}}$. This is just for notational convenience, and the set $\mathbb{N}^{\mathbb{Z}}$ can be replaced by $\mathbb{N}^{\mathbb{N}}$ taking a computable bijection between $\mathbb{N}$ and $\mathbb{Z}$.

Definition 5.12. Let G be a computable graph. We define $\mathscr{E}_2(G)$ as the set of all functions $x\colon \mathbb{Z}\to\mathbb{N}$ satisfying the following properties.

- For each even number n, G has a vertex $v_{x(n)}$ with index $x(n)$, and for each odd number m, G has an edge $e_{x(m)}$ with index $x(m)$.
- For each odd number n, the edge $e_{x(n+1)}$ joins $v_{x(n)}$ to $v_{x(n+2)}$.
- The graph homomorphism $T\colon [\![\mathbb{Z}]\!]\to G$ defined by $T(n)=v_{x(2n)}$, $T\{n,n+1\}=e_{x(2n+1)}$ is a two-sided infinite Eulerian trail on G.

The proof of the following result is very similar to the proof of Proposition 5.11, we just need to replace the algorithm for right-extensible trails by the algorithm for bi-extensible trails (Proposition 5.9).

Proposition 5.13. *Let G be a moderately computable graph satisfying $\mathscr{E}_2$. There is an algorithm which on input a finite sequence of natural numbers $(a_i)_{-n\leq i\leq n}$ decides whether the cylinder set*

$$\{x\in\mathbb{N}^{\mathbb{Z}}\mid x(-n)=a_{-n},\dots,x(n)=a_n\}$$

intersects $\mathscr{E}_2(G)$.

This finishes the proof of Theorem C.

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