UNDECIDABILITY OF DYNAMICAL PROPERTIES OF SFTS AND SOFIC SUBSHIFTS ON $\mathbb{Z}^2$ AND OTHER GROUPS

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Abstract. We study the algorithmic undecidability of abstract dynamical properties for sofic $\mathbb{Z}^2$-subshifts and subshifts of finite type (SFTs) on $\mathbb{Z}^2$. Within the class of sofic $\mathbb{Z}^2$-subshifts, we prove the undecidability of every nontrivial dynamical property. We show that although this is not the case for $\mathbb{Z}^2$-SFTs, it is still possible to establish the undecidability of a large class of dynamical properties. This result is analogous to the Adian-Rabin undecidability theorem for group properties. Besides dynamical properties, we consider dynamical invariants of $\mathbb{Z}^2$-SFTs taking values in partially ordered sets. It is well known that the topological entropy of a $\mathbb{Z}^2$-SFT can not be effectively computed from an SFT presentation. We prove a generalization of this result to *every* dynamical invariant which is nonincreasing by factor maps, and satisfies a mild additional technical condition. Our results are also valid for $\mathbb{Z}^d$, $d \geq 2$, and more generally for any group where determining whether a subshift of finite type is empty is undecidable.

1. Introduction

Informally, a $\mathbb{Z}^2$-subshift of finite type ($\mathbb{Z}^2$-SFT for short) is a set of colorings of $\mathbb{Z}^2$. This set is determined by a finite set of colors, and a finite set of local rules. These objects arise in different mathematical contexts, including first order logic [54], second order logic [38, 29], thermodynamic formalism [51], limit sets associated to cellular automata [19], and the study of tilings of the plane subject to matching rules and substitutions [46, 30].

Here we adopt the point of view of topological dynamics -the study of continous group actions on compact spaces- and more specifically symbolic dynamics. This is the study of continous and expansive group actions on compact spaces with topological dimension zero. Up to topological conjugacy, these dynamical systems are known as shift spaces or subshifts. A $\mathbb{Z}^2$-SFT is a particular type of subshift. In this case, the action of $\mathbb{Z}^2$ is that of translations, and the topology is the prodiscrete topology. We will also consider *sofic* $\mathbb{Z}^2$-subshifts. This means a $\mathbb{Z}^2$-subshift which is the image of a $\mathbb{Z}^2$-SFT by a surjective morphism of dynamical systems. These notions extend from $\mathbb{Z}^2$ to any countable group.

There exist several purely dynamical questions about $\mathbb{Z}^2$-SFTs whose answers have been enabled by recursion theory [40, 35, 24, 9]. For instance, the problem of classifying those real numbers which are the topological entropy, the entropy dimension, and the polynomial growth-rate of some $\mathbb{Z}^2$-SFT. In each case, the corresponding class of real numbers is characterized by a recursion-theoretical property [36, 45].

These results are related to the fact that $\mathbb{Z}^2$-SFTs can behave as models of universal computation. This means that it is possible to turn a computer program into a $\mathbb{Z}^2$-SFT in such a manner that algorithmic properties of the computer program

2020 *Mathematics Subject Classification.* 37B10, 37M99, 03D30, 68Q17.

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are translated to dynamical properties of the $\mathbb{Z}^2$-SFT. We use the terms algorithm and computer program as synonyms of the formal object of Turing machine.

The fact that $\mathbb{Z}^2$-SFTs can behave as models of computation can be regarded as an obstruction to perform computations or simulations on them. That is, algorithmically undecidable questions about computer programs are translated to algorithmically undecidable questions about $\mathbb{Z}^2$-SFTs. The fundamental result is Berger's theorem, which asserts that the *emptiness problem* for $\mathbb{Z}^2$-SFTs is algorithmically undecidable.

Theorem (Berger, [15])**.** *There exists no algorithm which given a $\mathbb{Z}^2$-SFT presentation (a finite alphabet $A \subset \mathbb{N}$ and a finite set of local rules) decides whether the associated SFT is empty or not.*

Let us recall that **dynamical properties** are those properties preserved by topological conjugacy, the notion of isomorphism in topological dynamics. In this article we are interested in understanding which dynamical properties of $\mathbb{Z}^2$-SFTs and of sofic $\mathbb{Z}^2$-subshifts can be detected algorithmically from a finite presentation of the system. Many such properties are known to be algorithmically undecidable, and indeed Lind coined the term "swamp of undecidability" to reflect this situation [43]. This metaphor naturally raises the following question.

Question. *Is every nontrivial dynamical property for $\mathbb{Z}^2$-SFTs (respectively, sofic $\mathbb{Z}^2$-subshifts) algorithmically undecidable?*

A property is **nontrivial** for a class of objects $\mathscr{C}$ if some element in $\mathscr{C}$ has the property, and some element in $\mathscr{C}$ fails to have the property.

In recursion theory, Rice's theorem asserts that *every* nontrivial question about the behaviour of computer programs is algorithmically undecidable [50]. This result can be proved by a reduction to the Halting problem. This is a fundamental and algorithmically undecidable problem in recursion theory [53].

Rice's theorem has been paradigmatic. Analogous results have been discovered in a variety of mathematical contexts, sometimes called Rice-like theorems [41, 22, 44, 34, 42, 33, 26, 23, 50, 1, 49]. Let us highlight here the Adian-Rabin theorem [2, 49], which is the "Rice-like theorem in group theory". The Adian-Rabin theorem states that all Markov properties are algorithmically undecidable from finite group presentations. A Markov property is a group property $\mathscr{P}$ for which there are two finitely presented groups G_+ and G_- satisfying the following:

- G_+ satisfies $\mathscr{P}$.
- Every finitely presented group where G_- embeds fails to satisfy $\mathscr{P}$.

The proof of the Adian-Rabin theorem goes by showing that a decidable Markov property could be used to solve the word problem of any finitely presented group. This contradicts the existence of finitely presented groups with undecidable word problem, a classic result of Novikov and Boone [47, 16].

In the following section we present our results. Among them is a result similar to Rice's theorem for sofic $\mathbb{Z}^2$-subshifts, and a result similar to the Adian-Rabin theorem for $\mathbb{Z}^2$-SFTs. In both cases the proof is by a reduction to the emptiness problem for $\mathbb{Z}^2$-SFTs. This problem is algorithmically undecidable by Berger's theorem.

2. Results

We will use standard terminology from topological dynamics, and in particular shift spaces on $\mathbb{Z}^2$ (see Section 3). A dynamical property is said to be decidable for $\mathbb{Z}^2$-SFTs when there exists an algorithm which, provided with the presentation of a $\mathbb{Z}^2$-SFT (a finite alphabet $A \subset \mathbb{N}$ and a finite set of local rules), decides whether the corresponding $\mathbb{Z}^2$-SFT satisfies the property. Otherwise, the property is said to be undecidable. We follow the same convention for sofic $\mathbb{Z}^2$-subshifts, in which case the presentation also specifies a topological factor map by a local function.

2.1. **Undecidability results for dynamical properties of sofic $\mathbb{Z}^2$-subshifts and $\mathbb{Z}^2$-SFTs.** Our main result regarding dynamical properties of sofic $\mathbb{Z}^2$-subshifts is that every single nontrivial dynamical property is undecidable.

Theorem 2.1. *Every nontrivial dynamical property for sofic $\mathbb{Z}^2$-subshifts is undecidable.*

This settles the "swamp of undecidability" for sofic $\mathbb{Z}^2$-subshifts with a simple answer, similar to Rice's theorem for computer programs. We will see that in the class of $\mathbb{Z}^2$-SFTs, the situation is slightly more complex, and not all nontrivial dynamical properties are undecidable.

Proposition 2.2. *The dynamical property of having at least one fixed point is decidable for $\mathbb{Z}^2$-SFTs.*

The proof of this result is straightforward. It follows that a Rice-like theorem is not possible for dynamical properties of SFTs on $\mathbb{Z}^2$. However, we can still prove the undecidability of a large classes of properties, which resemble a dynamical version of Markov properties for groups. In view of the strong analogy, we will use the term Berger property.

Definition 2.3. A dynamical property $\mathscr{P}$ of $\mathbb{Z}^2$-SFTs is called a **Berger property** if there are two $\mathbb{Z}^2$-SFTs X_- and X_+ satisfying the following conditions:

(1) X_+ satisfies $\mathscr{P}$.
(2) Every $\mathbb{Z}^2$-SFT which factors onto X_- fails to satisfy $\mathscr{P}$.
(3) There is a topological morphism from X_+ to X_-.

Our main result regarding the undecidability of dynamical properties of $\mathbb{Z}^2$-SFTs is that all Berger properties are undecidable.

Theorem 2.4. *Every Berger property of $\mathbb{Z}^2$-SFTs is undecidable.*

Let us observe that in the definition of Berger property, the set X_+ may be a subsystem of X_-, as a subshift inclusion is in particular a topological morphism. The subshift X_+ is also allowed to be the *empty subshift.* Some authors consider the empty set as a subshift [21], while others exclude it by definition [3]. Here we follow the first convention. This is rather natural: when we say that an algorithm is able to detect a property from $\mathbb{Z}^2$-SFT presentations, we understand that it can take as input a presentation of the empty subshift. Moreover, any presentation of the empty subshift should produce the same answer. We recall that infinitely many SFT presentations give rise to the empty subshift, and it follows from Berger's theorem that we cannot exclude these presentations in a computable manner.

A consequence of our convention is that a dynamical property must assign yes/no value to the empty subshift. Keeping this in mind, we have the following simple corollary of Theorem 2.4.

Corollary 2.5. *Every nontrivial dynamical property for SFTs which is preserved to topological factors (resp. extensions), and which is satisfied (resp. not satisfied) by the empty subshift, is undecidable.*

Whether we consider the empty subshift to have or not certain property does not seem relevant for dynamical purposes. However, this may have some effect when we consider the decidability of the property from presentations. For instance, the property of having no fixed point is decidable (Proposition 2.2), while the property of having no fixed point *and* being nonempty is undecidable. This follows from Corollary 2.5, as having no fixed point is preserved to topological extensions. However, this situation seems rather exceptional, and the results presented here can be used to prove that many dynamical properties of $\mathbb{Z}^2$-SFTs are undecidable, no matter which value is given to the empty subshift.

As examples of direct applications of our results, we consider the following dynamical properties of $\mathbb{Z}^2$-SFTs: being transitive, being minimal, and having topologically complete positive entropy (TCPE). We verify that these properties are undecidable, and that this remains unchanged if we allow or exclude the empty subshift from the property.

Example. A $\mathbb{Z}^2$-SFT is topologically transitive when it contains an element with dense orbit. This property is preserved to topological factors.

The undecidability of "being transitive or empty" follows directly from Corollary 2.5: this property is inherited to topological factors, and it is satisfied by the empty subshift.

The undecidability of "being transitive and nonempty" follows from Theorem 2.4, being a Berger property. This can be seen by taking X_- as a subshift with exactly two points (its topological extensions are not transitive), and $X_+ \subset X_-$ as a subsystem with a single point (transitive and nonempty).

Example. A $\mathbb{Z}^2$-SFT is minimal when it has no proper nonempty subsystem. The same reasoning as in the previous example shows that both "being minimal or empty" and "being minimal and nonempty" are undecidable properties.

Example. A $\mathbb{Z}^2$-SFT has topologically complete positive entropy (TCPE) when every topological factor is either a singleton with the trivial action by $\mathbb{Z}^2$, or has positive topological entropy. This property is inherited to topological factors.

The undecidability of "TCPE and nonempty" follows from Theorem 2.4, being a Berger property. In order to see this, let X_- be the $\mathbb{Z}^2$-SFT $\{0,1\}^{\mathbb{Z}^2} \cup \{2,3\}^{\mathbb{Z}^2}$. This system fails to have TCPE because it factors onto the SFT with exactly two configurations and zero topological entropy. Now let $X_+ \subset X_-$ be the $\mathbb{Z}^2$-SFT $\{0,1\}^{\mathbb{Z}^2}$. It is well known that the SFT $\{0,1\}^{\mathbb{Z}^2}$ has TCPE, so our claim that "TCPE and nonempty" is a Berger property follows. On the other hand, the undecidability of "TCPE or empty" follows directly from Corollary 2.5. The exact complexity of the property TCPE is computed in [55], see also Section 6.

The class of properties where these results apply is quite large, but the precise frontier between decidable and undecidable in the class of dynamical properties for $\mathbb{Z}^2$-SFTs seems rather complex. This topic is discussed in Section 6, where we observe that the set of all dynamical properties can be endowed with a pre-order relation, and that our undecidability proofs are realizations of this relation.

Now we move on to consider the computability of dynamical invariants of $\mathbb{Z}^2$-SFTs.

2.2. **Uncomputability results for dynamical invariants.** Topological entropy is a fundamental invariant in topological dynamics. The problem of computing this invariant for $\mathbb{Z}^2$-SFTs has been extensively studied, both for practical and theoretical purposes. There are classes of SFTs for which this problem becomes tractable [27, 48, 28, 36, 25]. In the general case, however, it is not possible to compute the topological entropy of a $\mathbb{Z}^2$-SFT from a presentation [36].

We provide a wide generalization of this phenomenon to abstract dynamical invariants. We show that every dynamical invariant which is nonincreasing by factor maps and satisfies a technical condition, can not be effectively computed from a $\mathbb{Z}^2$-SFT presentation, even with the promise that the input is the presentation of a nonempty subshift.

Theorem 2.6. *Let $\mathcal{I}$ be a dynamical invariant for $\mathbb{Z}^2$-SFTs taking values in $\mathbb{R}$ which is nonincreasing by factor maps, and for which there are two nonempty $\mathbb{Z}^2$-SFTs $X_- \subset X_+$ such that $\mathcal{I}(X_-) < \mathcal{I}(X_+)$.*

Then there exists no algorithm which on input the presentation of a nonempty $\mathbb{Z}^2$-SFT X and a rational number $\varepsilon > 0$, outputs a rational number whose distance to $\mathcal{I}(X)$ is at most ε.

It was proved in [36] that there are $\mathbb{Z}^2$-SFTs whose topological entropy is a non-computable real number[1], so in particular an algorithm as in the statement can not exist. Our result has a weaker conclusion, but it is much more general and its proof is much simpler. Although this result is not a direct consequence of our results for dynamical properties, it is proved with similar methods.

We now state a result for more general invariants taking values on partially ordered sets. This result is obtained from Corollary 2.5. A similar result was proved in [23] for the invariant of topological entropy of sets of tilings of the plane.

Theorem 2.7. *Let $\mathcal{I}$ be a dynamical invariant for $\mathbb{Z}^2$-SFTs taking values in a partially ordered set $(\mathscr{R}, \leq)$, which is nonincreasing by factor maps, and whose value is minimal on the empty subshift. Then for every $r \in \mathscr{R}$ the following properties of a $\mathbb{Z}^2$-SFT X are either trivial or undecidable:*

(1) $\mathcal{I}(X) \geq r$
(2) $\mathcal{I}(X) \leq r$
(3) $\mathcal{I}(X) > r$
(4) $\mathcal{I}(X) < r$

This result can be applied to dynamical invariants whose values are not real numbers. This includes, for example, the growth type or growth order of the pattern complexity function (in the sense of an equivalence class of functions), and recursion-theoretical invariants such as the Turing degree of the language of a $\mathbb{Z}^2$-SFT [39], the Muchnik degree, and Medvedev degree of the $\mathbb{Z}^2$-SFT [52].

2.3. **Extension of these results to subshifts on other groups.** Now we consider SFTs and sofic subshifts on groups different than $\mathbb{Z}^2$. Shift spaces on different groups have been widely studied in recent years. In this context, the undecidability of the emptiness problem for SFTs -also called *domino problem* in the literature- has been extended to a large class of groups. This includes $\mathbb{Z}^d$ for $d \geq 2$, all non virtually free groups with polynomial growth [11], classes of Baumslag-solitar groups [8, 7, 10], all non virtually free hyperbolic groups [14], and others [21, 37, 31, 5, 13, 12]. Indeed, it has been conjectured in [11] that the emptiness problem of SFTs is undecidable on every group which is non virtually free. A survey on the topic can be found in [4].

Our proofs are only based on the undecidability of the emptiness problem for $\mathbb{Z}^2$-SFTs, and all the results presented here admit the following natural generalization.

Theorem 2.8. *All the results stated in this section still hold if we replace $\mathbb{Z}^2$ by a finitely generated group G where the emptiness problem for subshifts of finite type is undecidable.*

When we replace $\mathbb{Z}^2$ by a group G as in the statement, the proofs remain almost unchanged. Some subtleties arise in relation to presentations of subshifts of finite type, factor maps, and sofic subshifts. This comes from the possibility that the word problem of the group G may be undecidable. Nevertheless, this is easily handled with the formalism introduced in [6].

[1] A real number r is computable if there exists an algorithm which on input a rational number $\varepsilon > 0$, outputs a rational number whose distance to r is at most ε. This notion was introduced by Turing in [53], see [17] for a modern treatment.

2.4. **Previously known results.** Let us review some related results which are present in the literature, these are mainly focused on sets of tilings of $\mathbb{Z}^2$. A tile is a square with colored edges of length 1. A tilieset τ is a finite set of tiles, and a tiling is a function $\mathbb{Z}^2 \to \tau$ which respects the rule that adjacent squares must share edges with the same color. A set of tilings of $\mathbb{Z}^2$ is the set of all tilings $\mathbb{Z}^2 \to \tau$, for some tileset τ.

Sets of tilings of $\mathbb{Z}^2$ and $\mathbb{Z}^2$-SFTs are closely related. Every set of tilings of $\mathbb{Z}^2$ is a $\mathbb{Z}^2$-SFT. On the other hand, every $\mathbb{Z}^2$-SFT is topologically conjugate to at least one set of tilings (where both are endowed with the $\mathbb{Z}^2$ action of translations). However, isomorphism notions for sets of tilings of $\mathbb{Z}^2$ do not correspond to topological conjugacy of $\mathbb{Z}^2$-SFTs.

In the article [20] the authors prove the undecidability of the set equality and the set inclusion for sets of tilings of $\mathbb{Z}^2$ (and indeed also for $\mathbb{Z}^2$-SFTs), this as a "first step towards a Rice theorem for tilings".

In the article [42] the authors introduce a relation $\prec$ for tilesets which can be imagined as a "change of resolution". Loosely speaking we write $\tau \prec \tau'$ if every tiling produced by τ can be uniquely sliced in rectangular $n \times m$ blocks formed by tiles, and these blocks behave like tiles from τ'. The authors proved the undecidability of every nontrivial property of sets of tilings of $\mathbb{Z}^2$ which is preserved by "change of resolution", and even a Kleene-like fixed point theorem asserting that for any computable function on $\mathbb{N}$ there is an index of a tileset which is a fixed point "up to change of resolution". The proof of these results relies strongly on the particular structure of $\mathbb{Z}^2$.

In the article [23] Delvenne and Blondel prove some results for sets of tilings and Turing machines, but regarded as dynamical systems. For instance, the authors prove that every property of sets of tilings which is not satisfied by the empty set and which is preserved by direct products among nonempty systems is undecidable. This result is applied (with a rather different argument that ours), to prove a result similar to Theorem 2.7 for the invariant of topological entropy of sets of tilings.

Our results may be seen as a continuation of Delvenne and Blondel's results, but in the terminology of shift spaces. In order to provide a complete collection of results, we also prove a sligthly stronger version of their result for $\mathbb{Z}^2$-SFTs.

Theorem 2.9. *Let $\mathscr{P}$ be a dynamical property of $\mathbb{Z}^2$-SFTs satisfying the following:*

(1) *There is an SFT X_+ such that for every nonempty SFT X, the direct product $X \times X_+$ has the property $\mathscr{P}$.*
(2) *The empty SFT does not satisfy the property $\mathscr{P}$.*

Then $\mathscr{P}$ is undecidable.

Despite our framework is different, we remark that the proof presented here follows the same key ideas as Delvenne and Blondel's proof. The generalization of this result to other groups as in Theorem 2.8 is also implicit in [23], and we do not take credit for it.

Paper structure. In Section 3 we review some background on topological dynamics, shift spaces, and recursion theory. In Section 4 we prove our results for sofic $\mathbb{Z}^2$-subshifts and $\mathbb{Z}^2$-SFTs. In Section 5 we prove Theorem 2.8. This is done by explaining how to modify the proofs given for $\mathbb{Z}^2$ in Section 4.

In Section 6 we provide some examples which show that the hypotheses of some of our results are necessary. We also observe that the set of dynamical properties of SFTs and sofic subshifts can be given certain pre-order relation, and that all our undecidability proofs are realization of this relation.

Acknowledgements. The author is grateful to his advisors Sebastián Barbieri and Cristóbal Rojas for their continous support and advise, and to Mathieu Sablik for discussions regarding this work in the context of a research stay. The author is also grateful to Valentino Delle Rose, Jaime Gomez, and Paola Burgos for their helpful comments on this work.

This research was partially supported by ANID 21201185 doctorado nacional, ANID/Basal National Center for Artificial Intelligence CENIA FB210017, and the European Union's Horizon 2020 research and innovation program under the Marie Sklodowska-Curie grant agreement No 731143.

3. Background

We will use standard terminology from topological dynamics and in particular shift spaces. The reader is referred to [19].

Topological dynamics. A $\mathbb{Z}^2$-topological dynamical system is a pair (X, T) of a compact metrizable space X and a continous left group action $T\colon \mathbb{Z}^2 \times X \to X$. We may write $\mathbb{Z}^2 \curvearrowright X$ for short. A morphism of dynamical systems $\mathbb{Z}^2 \curvearrowright X$, $\mathbb{Z}^2 \curvearrowright Y$ is a continous map $\phi\colon X \to Y$ which conmutes with the corresponding group actions. A morphism of dynamical systems which is surjective (resp. injective, resp. bijective) is called a topological factor (resp. embedding, resp. conjugacy). When there exists a topological factor $\phi\colon X \to Y$ we also say that Y is a topological factor of X, X is a topological extension of Y, and that X factors over Y. The direct product of two $\mathbb{Z}^2$-dynamical systems $\mathbb{Z}^2 \curvearrowright X$, $\mathbb{Z}^2 \curvearrowright Y$ is given by the componentwise action on the product space $X \times Y$. A subsystem of a topological dynamical system $\mathbb{Z}^2 \curvearrowright X$ is a subset $Y \subset X$ which is topologically closed and invariant under the action. The disjoint union of $\mathbb{Z}^2$-topological dynamical systems (X, T) and (Y, S) is $(X \times \{0\} \cup Y \times \{1\}, R)$, where the action R is defined by requiring $x \mapsto (x, 0)$ and $y \mapsto (y, 1)$ to be topological embeddings from X and Y to $X \times \{0\} \cup Y \times \{1\}$.

Shift spaces on $\mathbb{Z}^2$. Let A be a finite set. We endow the set $A^{\mathbb{Z}^2} = \{x\colon \mathbb{Z}^2 \to A\}$ with the product of the discrete topologies, and the left and continous action of $\mathbb{Z}^2 \curvearrowright A^{\mathbb{Z}^2}$ by translations. This action is defined by the expression $(\boldsymbol{n}x)(\boldsymbol{m}) = x(\boldsymbol{m} - \boldsymbol{n})$, $\boldsymbol{n}, \boldsymbol{m} \in \mathbb{Z}^2$. A **configuration** is an element $x \in A^{\mathbb{Z}^2}$. A **pattern** is a function $p\colon S \to A$, where S is a finite subset of $\mathbb{Z}^2$. A pattern $p\colon S \to A$ **appears** in a configuration x if for some $\boldsymbol{n} \in \mathbb{Z}^2$ the restriction of $\boldsymbol{n}x$ to S equals p. A **$\mathbb{Z}^2$-subshift**, or **subshift on** $\mathbb{Z}^2$, is a topologically closed and translation-invariant subset of $A^{\mathbb{Z}^2}$, where A is a finite set. In this case A is called the **alphabet** of the subshift.

A $\mathbb{Z}^2$-subshift X is **of finite type**, abreviated SFT, if there is an alphabet A and a finite set of patterns $\mathcal{F}$ such that X is the set of all configurations in $A^{\mathbb{Z}^2}$ in which no pattern of $\mathcal{F}$ appears. In this situation we say that X was obtained by **forbidding** the patterns in $\mathcal{F}$.

A $\mathbb{Z}^2$-subshift is **sofic** if it is the topological factor of a $\mathbb{Z}^2$-SFT. By the Curtis-Hedlund-Lyndon theorem such a factor map must be the restriction of a **sliding block code,** which is is a function $\phi\colon A^{\mathbb{Z}^2} \to B^{\mathbb{Z}^2}$ satisfying the following property: there is a finite set $S \subset \mathbb{Z}^2$ and a function $\mu\colon A^S \to B$ such that for every x and $\boldsymbol{n} \in \mathbb{Z}^2$ we have $\phi(x)(\boldsymbol{n}) = \mu((\boldsymbol{n}x)\mid_S)$. In this situation μ is called a **local function** for ϕ.

Remark 3.1. With these definitions, the disjoint union of two subshifts and the direct product of two subshifts are **not** subshifts. However, they are dynamical systems topologically conjugate to subshifts.

Recursion theory. We will need very few concepts from computability theory. A function $f\colon \mathbb{N} \to \mathbb{N}$ is **computable** if there exists a Turing machine which on input n outputs $f(n)$. We use the word **algorithm** to refer to the formal object of Turing machine. A set $N \subset \mathbb{N}$ is **decidable** if its characteristic function is computable. In this article we will not consider partial functions.

Remark 3.2. Along this paper we will need to perform computations over finite objects which are not natural numbers, in which case we assume that they are represented natural numbers in a canonical way. This is the case for tuples such as $\mathbb{N}^2$, the set of words T^* over a finite alphabet T, as well as finite subsets of $\mathbb{N}$, finite sets of patterns, local functions, and presentations to be defined later. A completely formal treatment of these computations can be done using numberings, but we avoid this level of detail as this it is completely standard in computability theory. See for instance [32, Chapter 14].

4. (Un)decidability results for $\mathbb{Z}^2$

In this section we prove our results regarding $\mathbb{Z}^2$-SFTs and sofic $\mathbb{Z}^2$-subshifts. We will define presentations precisely, and prove some elemental results about these presentations.

4.1. **SFT presentations and indices.** We define an $\mathbb{Z}^2$**-SFT presentation** as a pair $(A, \mathcal{F})$ of a finite subset $A \subset \mathbb{N}$, and a finite set of patterns $\mathcal{F}$ on alphabet A. The subshift of finite type associated to the presentation $(A, \mathcal{F})$ is the set

$$X_{(A,\mathcal{F})} = \{x\colon \mathbb{Z}^2 \to A \mid \text{no pattern of } \mathcal{F} \text{ appears in } x\}.$$

We now introduce a formal device which will be used in our proofs. We associate to each presentation $(A, \mathcal{F})$ a natural number which contains the finite information of the presentation. This number is called the **index** of the presentation. We require this indexing of all presentations to satisfy the following property: from the index n of the presentation $(A, \mathcal{F})$, we can computably recover the sets A and $\mathcal{F}$, and viceversa. It is a standard fact that such a numbering exists (see Remark 3.2).

Observe that then natural numbers are in bijective correspondence with all $\mathbb{Z}^2$-SFT presentations. If n is the index of the presentation $(A, \mathcal{F})$, then we define X_n as the subshift $X_{(A,\mathcal{F})}$, and we also say that n is an index for X_n. If n is the index of $(A, \mathcal{F})$, we denote by A_n the alphabet A, and by $\mathcal{F}_n$ the set of forbidden patterns $\mathcal{F}$.

We now prove two basic results regarding products and disjoint unions of SFTs. We fix for the rest of this section a computable bijection $\alpha\colon \mathbb{N}^2 \to \mathbb{N}$, and computable functions $\pi_1, \pi_2\colon \mathbb{N} \to \mathbb{N}$ defined by the expressions $\pi_i(\alpha(n_1, n_2)) = n_i$, $i = 1, 2$. These functions will be use to "simulate" products and projections of alphabets, as we need our alphabets to be finite subsets of $\mathbb{N}$.

Lemma 4.1. *There is a computable function $f\colon \mathbb{N}^2 \to \mathbb{N}$ that on input (n, m), outputs the index of a $\mathbb{Z}^2$-SFT which is topologically conjugate to the product $X_n \times X_m$.*

Proof. Let A_n and A_m be the alphabets of X_n and X_m, with sets of forbidden patterns $\mathcal{F}_n$ and $\mathcal{F}_m$ respectively. We just need to use the function α to replace the alphabet $A_n \times A_m$ by a subset of $\mathbb{N}$, and transfer the forbidden patterns $\mathcal{F}_n$ and $\mathcal{F}_m$ to this new alphabet.

On input (n, m) our algorithm proceeds as follows. First, the function α maps $A_n \times A_m$ bijectively to the set $B = \{\alpha(a, b) \mid a \in A_n, b \in A_m\} \subset \mathbb{N}$, which is the alphabet of the new SFT. Then we compute a set $\mathcal{F}$ of forbidden patterns on alphabet B as follows. For each pattern $p\colon S \to A_n$ in $\mathcal{F}_n$, we add to $\mathcal{F}$ every pattern $q\colon S \to B$ such that $\pi_1 \circ q = p$. In a similar manner for each $q\colon S \to A_m$

in $\mathcal{F}_m$, we add to $\mathcal{F}$ every pattern $q\colon S \to B$ such that $\pi_2 \circ q = p$. This process is computable because α, π_1 and π_2 are computable functions, so on input (n,m) we can compute the alphabet B and the set of patterns $\mathcal{F}$. We define $f(n,m)$ as the index of the presentation $(B,\mathcal{F})$. By our hypothesis on the indexing of presentations, it follows that f is a computable function.

We now verify that $X_{f(n,m)}$ is topologically conjugate to the direct product $X_n \times X_m$. We define a map

$$\phi_{(n,m)}\colon X_{f(n,m)} \to X_n \times X_m$$

by $x \mapsto (\boldsymbol{n} \mapsto \pi_1(x(\boldsymbol{n})), (\boldsymbol{n} \mapsto \pi_2(y(\boldsymbol{n}))))$, $\boldsymbol{n} \in \mathbb{Z}^2$. It is straightforward that this map is a topological conjugacy. □

Lemma 4.2. *There is a computable function $u\colon \mathbb{N}^2 \to \mathbb{N}$ that on input (n,m), outputs the index of a $\mathbb{Z}^2$-SFT which is topologically conjugate to the disjoint union of X_n and X_m.*

Proof. The algorithm makes use of two values n_0 and n_1, which do not depend on the input. The value n_0 is an index of the $\mathbb{Z}^2$-SFT on alphabet $\{0\}$, and which only has the configuration with constant value 0. Similarly, n_1 is an index of the $\mathbb{Z}^2$-SFT on alphabet $\{1\}$, and which only has the configuration with constant value 1.

On input (n,m), the algorithm proceeds as follows. First, compute $f(n,n_0)$ and $f(m,n_1)$, where f is the function from Lemma 4.1. Let us recall that $X_{f(n,n_0)}$ has alphabet $A_{f(n,n_0)}$, and is defined by the set of forbidden patterns $\mathcal{F}_{f(n,n_0)}$. We let $B \subset \mathbb{N}$ be the union of the (disjoint) alphabets $A_{f(n,n_0)}$ and $A_{f(m,n_1)}$, and we let $\mathcal{F}$ be the union of the sets of forbidden patterns $\mathcal{F}_{f(n,n_0)}$ and $\mathcal{F}_{f(m,n_1)}$. Finally we define $u(n,m)$ as the index of the presentation $(B,\mathcal{F})$. By our hypothesis on the indexing of presentations and the fact that f is computable, it follows that u is a computable function.

We now verify that $X_{u(n,m)}$ is topologically conjugate to the disjoint union of X_n and X_m. Indeed, we just need no observe that the following map defines a topological conjugacy:

$$\begin{aligned} X_n \times \{0\} \cup X_m \times \{1\} &\to X_{u(n,m)} \\ (x,0) &\mapsto (\boldsymbol{n} \mapsto \alpha(x(\boldsymbol{n}),0)), \\ (x,1) &\mapsto (\boldsymbol{n} \mapsto \alpha(x(\boldsymbol{n}),1)), \quad \boldsymbol{n} \in \mathbb{Z}^2. \end{aligned}$$

□

4.2. **Undecidability results for dynamical properties of $\mathbb{Z}^2$-SFTs.** Here we prove our undecidability results for $\mathbb{Z}^2$-SFTs. Our proofs are based on Berger's theorem, which asserts the undecidability of the problem of determining whether X_n is empty, for $n \in \mathbb{N}$. We start with Theorem 2.4, which states that every Berger property is undecidable. We recall that Berger properties were defined in Definition 2.3.

Proof of Theorem 2.4. Let $\mathscr{P}$ be a Berger property, and let X_+, X_- be as in Definition 2.3. Let us first explain the proof idea. Given a $\mathbb{Z}^2$-SFT X_n, we let Z be the disjoint union of $X_n \times X_+$ and X_-. We now observe the following:

(1) If X_n is empty, then Z is topologically conjugate to X_-.
(2) If X_n is nonempty, then Z factors over X_+. This follows from two facts: that for X_n nonempty $X_n \times X_+$ factors over X_+, and that there is a topological morphism from X_- to X_+.

It follows that Z has property $\mathscr{P}$ if and only if X_n is nonempty. In order to prove our claim that $\mathscr{P}$ is undecidable it suffices to show that a presentation for a subshift topologically conjugate to Z can be computed from the index n. Indeed, if $\mathscr{P}$ was decidable, in input n we could compute an index for Z and use the decidability of $\mathscr{P}$ to decide whether $X_n \neq \emptyset$, contradicting Berger's theorem.

Let f be the function from Lemma 4.1, and let u be the function from Lemma 4.2. Let n_+ be an index for X_+, and let n_- be an index for X_-. It follows that the function g defined by $n \mapsto u(f(n, n_+), n_-)$ is computable. By construction, $X_{g(n)}$ is topologically conjugate to the disjoint union of $X_n \times X_+$ and X_-. By the explanation given above, this proves the undecidability of $\mathscr{P}$. $\square$

Now we prove Corollary 2.5. We recall that this result asserts the undecidability of every nontrivial dynamical property for SFTs which is preserved to topological factors (resp. extensions), and which is satisfied (resp. not satisfied) by the empty subshift.

Proof of Corollary 2.5. Let $\mathscr{P}$ be a nontrivial property which is inherited to topological factors and is satisfied by the empty subshift. Let X_+ be the empty subshift. As the property $\mathscr{P}$ is nontrivial, there is a $\mathbb{Z}^2$-SFT X_- which fails to satisfy $\mathscr{P}$. Then the SFTs X_- and X_+ show that $\mathscr{P}$ is a Berger property as in Definition 2.3. The undecidability of $\mathscr{P}$ follows from Theorem 2.4.

Now, let $\mathscr{P}$ be a nontrivial property which is inherited to topological extensions, and is not satisfied by the empty subshift. Then the property "not $\mathscr{P}$" is a nontrivial property which is inherited to topological factors, and is satisfied by the emtpy subshit. As "not $\mathscr{P}$" is undecidable, the property $\mathscr{P}$ is undecidable as well. $\square$

We now prove our last undecidability result for dynamical properties of $\mathbb{Z}^2$-SFTs, Theorem 2.9. This result asserts the undecidability of dynamical properties which satisfy certain condition with direct products (see Theorem 2.9).

Proof of Theorem 2.9. Let $\mathscr{P}$ be a property as in the statement, and let X_+ be an SFT as in the statement. Let f be the computable function defined in Lemma 4.1. Let n_+ be an index for X_+, and observe that for every $n \in \mathbb{N}$, the subshift $X_{f(n,n_+)}$ satisfies $\mathscr{P}$ if and only if X_n is nonempty.

If the property $\mathscr{P}$ was decidable, then we could decide whether a subshift X_n is empty by computing the index $f(n, n_+)$, and then checking whether $X_{f(n,n_+)}$ satisfies $\mathscr{P}$. As this contradicts Berger's theorem, it follows that $\mathscr{P}$ is undecidable. $\square$

4.3. **Undecidability results for dynamical invariants of $\mathbb{Z}^2$-SFTs.** Here we prove our results for dynamical invariants of $\mathbb{Z}^2$-SFTs. We start with Theorem 2.7, whose statement we recall now. Let $\mathcal{I}$ be a dynamical invariant for $\mathbb{Z}^2$-SFTs taking values in $\mathbb{R}$, which is nonincreasing by factor maps, and for which there are two nonempty SFTs $X_- \subset X_+$ with $\mathcal{I}(X_-) < \mathcal{I}(X_+)$. Then there is no algorithm which on input a presentation of a nonempty SFT X and a rational number ε, outputs a rational number whose distance to $\mathcal{I}(X)$ is at most ε.

Proof of Theorem 2.7. Let $\mathcal{I}$ be an invariant as in the statement, let $X_- \subset X_+$ be two SFTs as in the statement, and let q be a rational number such that $\mathcal{I}(X_-) < q < \mathcal{I}(X_+)$. Now let g be a computable function which on input n, outputs the index of an SFT which is topologicaly conjugate to the disjoint union of $X_n \times X_+$ and X_-. The existence of g is ensured by Lemma 4.1 and Lemma 4.2: we can let $g(n) = u(f(n, n_+), n_-)$, where n_+ and n_- are indices for X_+ and X_-, respectively.

Observe that $X_{g(n)}$ is always nonempty, this follows from the fact that X_- is nonempty. Moreover, when X_n is empty we have $\mathcal{I}(X_{g(n)}) = \mathcal{I}(X_-) < q$, and otherwise $\mathcal{I}(X_{g(n)}) \geq \mathcal{I}(X_+) > q$. The last inequality follows from the fact that for X_n nonempty, $X_{g(n)}$ factors over X_+.

Now we assume the existence of an algorithm as in the statement, and exhibit an algorithm which on input n decides whether $X_n \neq \emptyset$, a contradiction to Berger's theorem.

On input n, we proceed as follows. For each $s \in \mathbb{N}$ and in an ordered manner, we compute a rational number whose distance to $\mathcal{I}(X_{g(n)})$ is at most $1/s$. This is possible by hypothesis. For some s big enough, we will be sure that $\mathcal{I}(X_{g(n)})$ lies in an interval $\{r \in \mathbb{R} \mid a < r < b\}$ whose endpoints are rational numbers, and which is completely contained in either $\{r \in \mathbb{R} \mid r < q\}$ or $\{r \in \mathbb{R} \mid r > q\}$. When we reach this point we stop the search, and conclude as follows. If the interval is contained in $\{r \in \mathbb{R} \mid r < q\}$, we conclude that $\mathcal{I}(X_n) < q$ and thus X_n is empty. Otherwise we conclude that $\mathcal{I}(X_n) > q$, and thus X_n is nonempty.

Such a number s must exist: if $\frac{1}{s}$ is a lower bound to the distance between $\mathcal{I}(X_{g(n)})$ and q, then s must satisfy the mentioned condition. We have found the mentioned contradiction, so the proof is finished. □

We now prove Theorem 2.7, which asserts that if $\mathcal{I}$ is an invariant for $\mathbb{Z}^2$-SFTs taking values in the partially ordered set $(\mathscr{R}, \leq)$ which is nonincreasing by factor maps and attains a minimal value on the empty subshift, then for every $r \in \mathscr{R}$ the properties $\mathcal{I}(X) \geq r$, $\mathcal{I}(X) \leq r$, $\mathcal{I}(X) > r$, $\mathcal{I}(X) < r$ are either trivial or undecidable. The proof is based on Corollary 2.5.

Proof of Theorem 2.7. Let $\mathcal{I}$ be an invariant as in the statement, and let r be an element in the partially ordered set $(\mathscr{R}, \leq)$. Let us first prove our claim for the property $\mathcal{I}(X) \leq r$. We must consier two cases:

(1) If $\mathcal{I}(\emptyset) \leq r$, then the property $\mathcal{I}(X) \leq r$ is preserved to topological factors and is satisfied by the empty subshift. It follows from Corollary 2.5 that the property $\mathcal{I}(X) \leq r$ is either trivial or undecidable.
(2) If $\mathcal{I}(\emptyset) \leq r$ fails, then $\mathcal{I}(X) \leq r$ also fails for every $\mathbb{Z}^2$-SFT X. This follows from the fact that $\mathcal{I}(\emptyset) \leq \mathcal{I}(X)$, and the transitivity of $\leq$. It follows that then $\mathcal{I}(X) \leq r$ is a trivial property.

The proof of the claim for the property $\mathcal{I}(X) \geq r$ is similar: if $\mathcal{I}(\emptyset) \geq r$, then the claim follows from Corollary 2.5, otherwise the property is trivial. The proofs of the claims for strict inequalities are very similar, we just need to replace non strict inequalities by strict inequalities in the arguments. □

4.4. **Undecidability result for sofic $\mathbb{Z}^2$-subshifts.** Here we prove Theorem 2.1, which asserts that every nontrivial dynamical property for sofic $\mathbb{Z}^2$-subshifts is undecidable.

Let us first define the corresponding notion of presentation. A **sofic $\mathbb{Z}^2$-subshift presentation** is a tuple $(A, \mathcal{F}, \mu, B)$, where $(A, \mathcal{F})$ is a $\mathbb{Z}^2$-SFT presentation, $\mu\colon A^S \to B$ is a local function, $S \subset \mathbb{Z}^2$ is a finite set, and $B \subset \mathbb{N}$ is a finite set. The sofic subshift associated to this presentation $Y_{(A,\mathcal{F},\mu,B)}$ is defined as the image of $X_{(A,\mathcal{F})}$ under the topological factor map whose local function is μ.

As we did for SFTs we assign an index n to each tuple $(A, \mathcal{F}, \mu, B)$ in a manner such that from the index n we can computably recover $A, \mathcal{F}, \mu, B$ and vice versa (see Remark 3.2). As we did with SFTs we denote by Y_n the sofic $\mathbb{Z}^2$-subshift whose presentation is that of index n, and we also say that n is an index for Y_n. Let us remark that we keep the notations introduced for SFTs: X_n is the SFT with index n, A_n is its alphabet, and $\mathcal{F}_n$ is a set of defining forbidden patterns for X_n.

Proof of Theorem 2.1. Let $\mathscr{P}$ be a dynamical property which is nontrivial for sofic $\mathbb{Z}^2$-subshifts. Replacing $\mathscr{P}$ by its negation if necessary we can assume that the empty subshift does not satisfy the property $\mathscr{P}$. We also fix a sofic $\mathbb{Z}^2$-subshift Y_+ satisfying $\mathscr{P}$.

For the proof we construct a computable function $h\colon \mathbb{N} \to \mathbb{N}$ having the following property. If the SFT X_n is nonempty then $Y_{h(n)}$ is equal to Y_+, and otherwise $Y_{h(n)}$ is empty. Thus $Y_{h(n)}$ has property $\mathscr{P}$ if and only if X_n is nonempty.

The existence of the computable function h proves that the property $\mathscr{P}$ is undecidable. Indeed, if the property $\mathscr{P}$ was decidable, we could use it to decide whether X_n is empty by first computing the index $h(n)$ of the sofic $\mathbb{Z}^2$-subshift $Y_{h(n)}$, and then checking whether $Y_{h(n)}$ has property $\mathscr{P}$. This contradicts Berger's theorem.

Let us first explain the construction of the function h. Let X_+ be a nonempty $\mathbb{Z}^2$-SFT which factors over Y_+ via the topological factor map ϕ, and observe that for every $n \in \mathbb{N}$ the subshift $X_+ \times X_n$ is empty if and only if X_n is empty. If X_n is nonempty, then we can compose with the the projection to the first coordinate ρ_1 and define in this manner a topological factor map from $X_+ \times X_n$ to Y_+.

$$X_+ \times X_n \xrightarrow{\rho_1} X_+ \xrightarrow{\phi} Y_+$$

The sofic $\mathbb{Z}^2$-subshift defined as the image of $X_+ \times X_n$ under the topological factor map $\phi \circ \rho_1$ equals Y_+ whenever X_n is nonempty, and otherwise is the empty subshift. Our function h just reproduces this procedure, while keeping the alphabets as subsets of $\mathbb{N}$.

We provide now a precise definition of h, this shows that it is a computable function. We fix for the rest of this proof a presentation for the sofic $\mathbb{Z}^2$-subshift Y_+, denoted $(A_+, \mathcal{F}_+, \mu_+, B_+)$. Thus the subshift Y_+ has alphabet B_+ and there is a topological factor map $X_{(A_+,\mathcal{F}_+)} \to Y_+$ with local function $\mu_+ : A_+^S \to B_+$, where $S \subset \mathbb{Z}^2$ is a finite set. We also fix for the rest of the argument a natural number n_+ which is an index for the SFT presentation $(A_+, \mathcal{F}_+)$.

The procedure followed by h on input n is as follows. The first step is to compute $f(n, n_+)$, an index of a subshift topologically conjugate to $X_n \times X_{n_+}$ (f is the function from Lemma 4.1). Then define a local function $\mu_n\colon A^S_{f(n,n_+)} \to B_+$ by the following rule. On input $p\colon S \to A_{f(n,n_+)}$, the function μ_n outputs the value of μ_+ evaluated in the pattern $\pi_2 \circ p\colon S \to A_{n_+}$. Thus there is an effective manner to compute μ_n from the index n. We denote by ψ_n the topological factor map with local function μ_n. Finally, $h(n)$ is defined as the index of the presentation $(A_{f(n,n_+)}, \mathcal{F}_{f(n,n_+)}, \mu_n, B_+)$.

Observe that in this manner, $Y_{h(n)}$ is defined as the image of the subshift $X_{f(n,n_+)}$ under the factor map ψ_n. It is clear from its definition that h is a computable function. It is clear that whenever X_n is empty, $Y_{h(n)}$ is also empty. On the other hand if X_n is nonempty, then it is clear from the definitions that the following diagram conmutes.

$$\begin{tikzcd}
X_{f(n_+,n)} \arrow[d, "\phi_{(n_+,n)}"] \arrow[rrd, "\psi_n"] & & \\
X_+ \times X_n \arrow[r, "\rho_1"] & X_+ \arrow[r, "\phi"] & Y_+
\end{tikzcd}$$

FIGURE 1. Conmutative diagram for nonempty X_n. Here, every arrow corresponds to a topological factor map.

The map $\phi_{(n_+,n)}$ is a topological conjugacy defined in Lemma 4.1. This shows that $Y_{h(n)}$, which is the image of $X_{f(n_+,n)}$ under ψ_n, is equal to Y_+ as claimed. This completes the proof. $\square$

4.5. **A decidable property for $\mathbb{Z}^2$-SFTs.** Here we prove our only decidability result for $\mathbb{Z}^2$-SFTs, Proposition 2.2. This result asserts that the property "having at least one fixed point" is decidable for $\mathbb{Z}^2$-SFTs.

Let us first make some observations. Let $(A, \mathcal{F})$ be an SFT presentation, and for each $a \in A$ denote by $x_a\colon \mathbb{Z}^2 \to A$ the configuration with constant value a. Observe that fixed points in $X_{(A,\mathcal{F})}$ are configurations x_a for some $a \in A$. Moreover, it is clear from the definitions that x_a lies in $X_{(A,\mathcal{F})}$ if and only if $\mathcal{F}$ does not contain a pattern p whose image has constant value a. We are now ready to prove the result.

Proof of Proposition 2.2. The algorithm proceeds as follows. On input an index n for a presentation $(A, \mathcal{F})$, we just check whether for some $a \in A$, $\mathcal{F}$ does not contains a pattern with constant value a. If the answer is positive, then we conclude that $X_{(A,\mathcal{F})}$ has a fixed point. Otherwise, we conclude that $X_{(A,\mathcal{F})}$ does not have a fixed point. $\square$

5. (Un)decidability results for other groups than $\mathbb{Z}^2$

In this section we prove Theorem 2.8, namely, that all the results stated in Section 2 are also valid for G-SFTs and sofic G-subshifts if G is a finitely generated group with an undecidable emptiness problem for SFTs.

5.1. **An overview.** The only notion from Section 4 which requires to be modified is that of presentations. As we do not assume our group to have decidable word problem, we must specify group elements by words. This can be interpreted by saying that we need two "layers" of presentations, the first being words presenting group elements, the second being finite objects (functions and sets) defined with these words, which present SFTs, sofic subshifts, and factor maps.

The previous informal interpretation can be expressed more precisely in the terminology of numberings or naming systems [32, Chapter 14]. In practice, we just need to take all the presentations defined in Section 4, and replace $\mathbb{Z}^2$-elements by words. After doing this appropriately, the proofs given in Section 4 can be carried over with no modification.

We fix for the rest of this section a finitely generated group G, and a finite and symmetric set of generators $T \subset G$. We assume that G is not $\mathbb{Z}^2$ to avoid notational ambiguity. We will make use of the terminology for topological dynamics and subshifts introduced in the Section 3, where $G \curvearrowright A^G$ is now defined by the expression $(gx)(h) = x(g^{-1}h)$. The reader is also referred to [19].

5.2. **SFT presentations and indices.** In this subsection we define SFT presentations for G-SFTs. Note that our only assumption here is that G is finitely generated.

We will make use of the notion of pattern coding introduced in [6], see also [4]. A **pattern coding** c is a finite subset of $T^* \times \mathbb{N}$. A pattern coding c is **consistent** if for every pair of elements (w, a) and (w', a') in c such that w and w' are words corresponding to the same group element in G, we have $a = a'$. A consistent pattern coding can be associated to a pattern $p(c)\colon S \subset G \to A$ in an obvious manner.

We define an G-**SFT presentation** as a pair $(A, \mathcal{C})$ of a finite subset $A \subset \mathbb{N}$, and a finite set of pattern codings $\mathcal{C}$. The subshift of finite type associated to the presentation $(A, \mathcal{C})$ is the set $X_{(A,\mathcal{C})}$ of all elements $x \in A^G$ such that for every consistent pattern coding $c \in \mathcal{C}$, $p(c)$ does not appear in x.

We associate to each presentation $(A, \mathcal{C})$ a natural number n, which we call its **index**. We require this indexing of all presentations to satisfy the following property: from the index n of the presentation $(A, \mathcal{C})$ we can computably recover the sets A and $\mathcal{C}$, and viceversa. It is a standard fact that such a numbering exists (see Remark 3.2). Now we observe that a generalization of Lemma 4.1 and Lemma 4.2 to G-SFT presentations and indices is straightforward: all relevant computations occur at the level of alphabets.

The **emptiness problem for G-SFTs** is the problem of deciding whether the G-SFT $X_{(A,\mathcal{C})}$ is empty from a presentation $(A, \mathcal{C})$, or equivalently, from the index of a presentation. As mentioned in the introduction, the emptiness problem for SFTs is known to be undecidable on many groups. The reader is referred to [4].

5.3. **Undecidability results for SFTs.** Here we observe that all the undecidability results proved for $\mathbb{Z}^2$-SFTs are also valid for G-SFTs, as long as G is a finitely generated group with undecidable emptiness problem for SFTs.

A careful reading of the proofs of Theorem 2.4, Theorem 2.9, and Theorem 2.6 shows that we only made use of the computability of direct products and disjoint unions of SFTs at the level of indices, plus the undecidability of the emptiness problem. We have observed, after defining G-SFT presentations, that the computability of direct products and disjoint unions of SFTs at the level of indices also holds for G-SFTs. It follows that the same proofs can be applied to G, where we only need to replace the corresponding results regarding products and disjoint unions.

Theorem 2.4 was used to prove Corollary 2.5, which then was used to prove Theorem 2.7. A careful reading of the proofs shows that these deductions are also valid under the sole assumption that the emptiness problem for G-SFTs is undecidable.

5.4. **Sofic subshifts.** Here we prove that as long as the emptiness problem for G-SFTs is undecidable, every nontrivial dynamical property of sofic G-subshifts is undecidable. We start by defining what is a sofic G-subshift presentation.

Let us recall that a sofic G-subshift is, by definition, a G-subshift which is the topological factor of a G-SFT. By the Curtis-Hendlund-Lyndon theorem [19, Theorem 1.8.1], such a topological factor map must be the restriction of a sliding block code $\phi\colon A^G \to B^G$. This means that there is a local function $\mu\colon A^S \to B$, where S is a finite subset of G, and where $\phi(x)(g) = \mu((g^{-1}x)|_S)$. We define a **local function presentation** as a function $\mu\colon A^S \to B$, where S is a finite subset of T^*.

A local function presentation μ defines a local function μ_0 as follows. Let $\pi\colon T^* \to G$ be the function which sends a word $w \in T^*$ to the corresponding group element, and let S_0 be the image of S under π. We have a function $A^{S_0} \to A^S$ defined by the expression $p \mapsto p \circ \pi$. In this manner μ defines a local function $\mu_0 : A^{S_0} \to A$, $S_0 \subset G$, by the expression $\mu_0(p) = \mu(p \circ \pi)$. In this situation we say that μ is a **presentation** for μ_0.

Finally a **sofic G-subshift presentation** is a tuple $(A, \mathcal{C}, \mu, B)$ of two alphabets $A, B \subset \mathbb{N}$, a finite set of pattern codings $\mathcal{C}$, and a local function presentation $\mu\colon A^S \to B$, $S \subset T^*$. The sofic subshift associated to this presentation, denoted by $Y_{(A,\mathcal{C},\mu,B)}$, is the image of $X_{(A,\mathcal{C})}$ under the topological factor map whose local function is presented by μ.

Assuming that the emptiness problem of G is undecidable and using these definitions, the proof of Theorem 2.1 can be adapted to G in a straightforward manner.

5.5. **A decidable property for G-SFTs.** Here we prove our only decidability result for G-SFTs, that is, that the property of having at least one fixed point can be algorithmically detected from a G-SFT presentation. This is the generalization of Proposition 2.2 to G-SFTs. Note that our only assumption here is that G is finitely generated.

Let $(A, \mathcal{C})$ be a G-SFT presentation, and for each $a \in A$ denote by $x_a \colon G \to A$ the configuration with constant value $a \in A$. Fixed points in $X_{(A,\mathcal{C})}$ are configurations x_a for some $a \in A$.

Let us observe that the pattern codings which determine whether the subshift has the property under consideration, are automatically consistent. Indeed, a pattern coding $c = \{(w_1, a_1), \dots, (w_n, a_n)\}$ with $a_1 = a_2 = \cdots = a_k$ is automatically consistent. On the other hand, if c is a consistent pattern coding and $p(c)$ appears on a fixed point of A^G, then c must have the form $\{(w_1, a_1), \dots, (w_n, a_n)\}$ with $a_1 = a_2 = \cdots = a_k$, this is clear from the definition.

It follows that in order to decide whether the G-SFT with presentation $(A, \mathcal{C})$ has at least one fixed point, we just need to verify whether for some $a \in A$, the set $\mathcal{C}$ fails to contain a pattern coding $\{(w_1, a_1), \dots, (w_n, a_n)\}$ with $a = a_1 = a_2 = \cdots = a_k$. This is clearly a computable operation.

6. Some remarks

In this section we observe that our proofs can be associated to a mathematical structure which expresses the "swamp of undecidability" in a precise manner. Moreover, we provide some examples which show that in the results stated in Section 2, we can not omit certain hypotheses.

6.1. **The swamp of undecidability.** In rough terms, we will consider the collection $\mathtt{SW}(G)$ of dynamical properties of G-SFTs, and the collection $\mathtt{SW}(G, \text{sofic})$ of dynamical properties of sofic G-subshifts. We will endow these collections with a pre-order relation $\leq$, and observe that the proofs given in the previous sections are realizations of this relation. As we shall see, we have already proved that $(\mathtt{SW}(G, \text{sofic}), \leq)$ has two "minimal elements". This gives a precise picture of the undecidability of all nontrivial dynamical properties of sofic subshifts. The collection $(\mathtt{SW}(G), \leq)$ seems a bit more complex.

We fix a finitely generated group G, an indexing (X_n) of G-SFTs, and an indexing (Y_n) of sofic G-subshifts as described in Section 5. We denote by $\sim$ the relation of topological conjugacy.

We define $\mathtt{SW}(G)$ as the set of all subsets $P \subset \mathbb{N}$ with $P \neq \emptyset$, $P \neq \mathbb{N}$, and such that when $n \in P$ and $X_n \sim X_m$, we also have that $m \in P$. The set $\mathtt{SW}(G, \text{sofic})$ is defined in a similar manner. That is, $\mathtt{SW}(G, \text{sofic})$ is the set of all subsets $P \subset \mathbb{N}$ with $P \neq \emptyset, P \neq \mathbb{N}$, and such that when $n \in P$ and $Y_n \sim Y_m$, we also have that $m \in P$.

Let us recall the concept of *many-one* reduction, a stronger class of reductions than Turing reductions which is commonly used in recursion theory. Given two sets $N, M \subset \mathbb{N}$, we write $N \leq_m M$ if there is a computable function $f \colon \mathbb{N} \to \mathbb{N}$ such that $n \in N \iff f(n) \in M$. The function f is then called a many-one reduction from N to M. In all our undecidability proofs we constructed many-one reductions, but we have avoided the concept with the hope of reducing the background required. These reductions have the additional property that they respect topological conjugacy. For this reason we define a specialization of many-one reductions as follows.

Given $P, Q \in \mathtt{SW}(G)$, we write $P \leq Q$ if there is a many-one reduction f from P to Q, and such that $X_n \sim X_m \Rightarrow X_{f(m)} \sim X_{f(m)}$. Similarly, given $P, Q \in \mathtt{SW}(G, \text{sofic})$ we write $P \leq Q$ if there is a many-one reduction f from P to Q, and such that $Y_n \sim Y_m \Rightarrow Y_{f(m)} \sim Y_{f(m)}$. We commit a slight abuse of notation by using the same symbol $\leq$ in both $\mathtt{SW}(G)$ and $\mathtt{SW}(G, \text{sofic})$, but this will cause no ambiguity.

All the statements in Section 2 are naturally expressed as properties of the sets $(\mathtt{SW}(G, \text{sofic}), \leq)$ and $(\mathtt{SW}(G), \leq)$. Let us now be more precise.

Observe that the sets $\{n \in \mathbb{N} \mid Y_n = \emptyset\}$ and $\{n \in \mathbb{N} \mid Y_n = \emptyset\}$ lie in $\mathtt{SW}(G, \text{sofic})$. We claim every element $P \in \mathtt{SW}(G, \text{sofic})$, satisfies either $\{n \in \mathbb{N} \mid Y_n = \emptyset\} \leq P$ or $\{n \in \mathbb{N} \mid Y_n \neq \emptyset\} \leq P$. Indeed, a careful reading of the proof of Theorem 2.1 yields a computable function h which is a many-one reduction as claimed. When G has an emptiness problem for SFTs, both $\{n \in \mathbb{N} \mid Y_n = \emptyset\}$ and $\{n \in \mathbb{N} \mid Y_n = \emptyset\}$ are undecidable, and thus every element in $\mathtt{SW}(G, \text{sofic})$ is an undecidable subset of $\mathbb{N}$.

We now make a few observations about $(\mathtt{SW}(G), \leq)$. We know that $\mathtt{SW}(G)$ has computable elements (regardless of G!), but the relation between these elements seems unclear. On the other hand, if $P \in \mathtt{SW}(G)$ is the index set of a Berger property, then the proof of Theorem 2.4 shows that $\{n \in \mathbb{N} \mid X_n \neq \emptyset\} \leq P$.

The study of pre-ordered sets which stem from the comparison of mathematical problems is classic in recursion theory. A natural question regarding $\mathtt{SW}(G)$ and $\mathtt{SW}(G, \text{sofic})$ is that of the complexity of classical dynamical properties, and their relation with the group G. This question has been already considered in the literature. For instance, consider $\mathtt{nonperiodic}_G \in \mathtt{SW}(G)$ be the dynamical property of containing at least one configuration with infinite orbit. In [18] it is proved that $\mathtt{nonperiodic}_{\mathbb{Z}^2}$ is Π^0_1-complete for $d = 2$, while $\mathtt{nonperiodic}_{\mathbb{Z}^d}$ is Σ^1_1-complete for $d \geq 4$. The complexity of $\mathtt{nonperiodic}_{\mathbb{Z}^3}$ is left as a question in [18]. A more basic instance is the property of being an empty SFT or sofic subshift. This dynamical property is known to be at least as hard as the word problem of G. More precisely, given a generating set T of G, we have

$$\{w \in T^* \mid w \text{ corresponds to the identity element } 1_G\} \leq_m \{n \in \mathbb{N} \mid X_n = \emptyset\}.$$

This is proved in [4]. This gives a lower bound to the m-degree and the Turing degree of *all* dynamical properties in $\mathtt{SW}(G, \text{sofic})$. A third instance is [55], where it is proved that the property TCPE $\in \mathtt{SW}(\mathbb{Z}^2)$ mentioned in the introduction is indeed Π^1_1-complete.

6.2. **Examples.** We now provide an example of a (not too useful) computable dynamical invariant which is nonincreasing by factors. This shows that in Theorem 2.6 and in Theorem 2.7, it is not enough to assume that $\mathcal{I}$ is nonincreasing by factors and attains at least two values. The invariant $\mathcal{I}$ is defined as follows. $\mathcal{I}(X)$ is 1 when X has no fixed point, and otherwise $\mathcal{I}(X)$ is 0. Thus $\mathcal{I}$ takes values in $\{0, 1\} \subset \mathbb{R}$. Let us observe that $\mathcal{I}$ is nonincreasing by factors. Let Y and X be SFTs, where Y is a topological factor of X. We verify that $\mathcal{I}(X) \geq \mathcal{I}(Y)$. Indeed, if $\mathcal{I}(X) = 1$, then the nonicreasing condition must be satisfied. On the other hand if $\mathcal{I}(X) = 0$ and X has a fixed point, then the same holds for Y. It follows that $\mathcal{I}(Y) = 0$ and thus $\mathcal{I}(X) \geq \mathcal{I}(Y)$.

We also observe that the third condition in Definition 2.3 is necessary to prove Theorem 2.4. Indeed, consider the property "having no fixed point". We could take $X_+ = \emptyset$, and X_- as any SFT with no fixed point (its topological extensions have no fixed points). However, this property is decidable by Proposition 2.2.

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