Are all nontrivial dynamical properties of $\mathbb{Z}^2$-SFTs undecidable?

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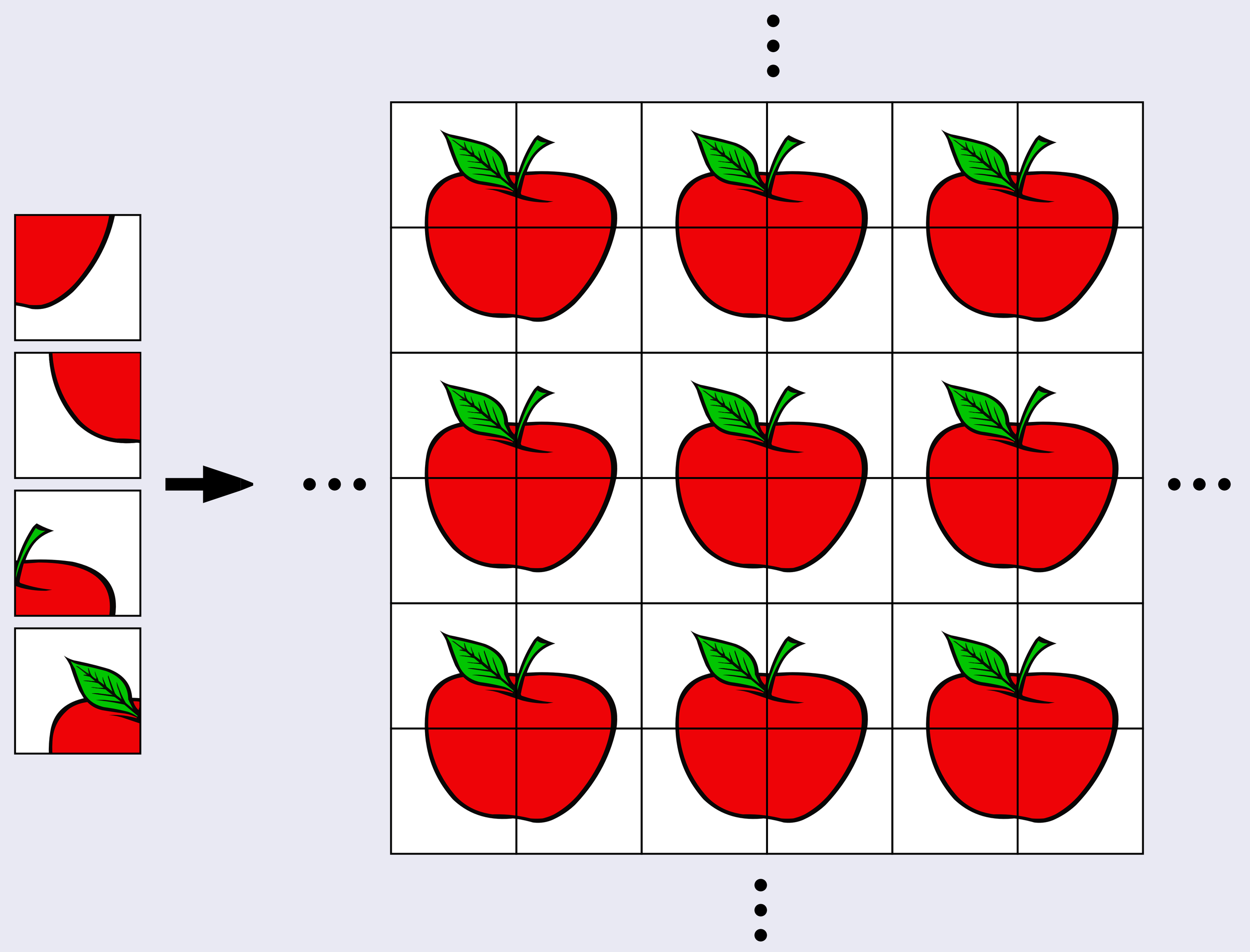
What are $\mathbb{Z}^2$-SFTs and sofic $\mathbb{Z}^2$-subshifts?

An example of $\mathbb{Z}^2$-SFT is the set of all tilings of $\mathbb{Z}^2$ obtained from a finite set of 1x1 square tiles with pictures.



In general, a $\mathbb{Z}^2$-**subshift** is a topologically closed subset of $\{x : \mathbb{Z}^2 \to A\}$ which is invariant by the $\mathbb{Z}^2$-action by translations. Here A is a finite set, which may contain square tiles, symbols, colors, etc. A subshift is said to be of **finite type** (SFT) when it is determined by a set of local rules. We see subshifts as topological dynamical systems, so we can talk about factor maps and morphisms. A $\mathbb{Z}^2$-subshift is **sofic** when it is the image of a $\mathbb{Z}^2$-SFT by a morphism of $\mathbb{Z}^2$-dynamical systems.

The swamp of undecidability

Many dynamical questions about $\mathbb{Z}^2$-SFTs, such as the existence of periodic and aperiodic points, are known to be undecidable. Lind used the term *swamp of undecidability* to reflect this situation. This metaphor raises the following question:

Question

Are all nontrivial dynamical properties of $\mathbb{Z}^2$-SFTs (resp. sofic subshifts) undecidable?

A property is trivial when it is satisfied by all SFTs or by no SFT. In this work we try to answer the question stated above, this leads us to explore the swamp of undecidability with tools from recursion theory.

The theorems of Rice and Adian-Rabin

The first result formalizing the idea of a *swamp of undecidability* in mathematics was Rice's theorem:

Theorem (Rice)

Every nontrivial question about the behaviour of computer programs is undecidable.

Similar results have been found in different mathematical contexts. We mention here the "Rice-like result in group theory".

Theorem (Adian and Rabin)

Let $\mathscr{P}$ be a group property such that there are two finitely presented groups G_- and G_+ satisfying the following conditions.

1. G_+ satisfies $\mathscr{P}$
2. Every group G in which G_- embeds fails to satisfy $\mathscr{P}$.

Then $\mathscr{P}$ is undecidable (from finite presentations of groups).

The properties as in the statement have been called **Markov** properties.

What we found out about dynamical properties

The following result expresses the *swamp of undecidability* for sofic subshifts in a precise manner.

Theorem ([1])

Every nontrivial dynamical property for $\mathbb{Z}^2$-sofic subshifts is undecidable.

For SFTs the situation is more complex, and a similar result is not possible: the property "having at least one fixed point" is a nontrivial dynamical property for $\mathbb{Z}^2$-SFTs, and it is decidable. However, we found a result which resembles the Adian-Rabin theorem:

Theorem ([1])

Let $\mathscr{P}$ be a dynamical property of $\mathbb{Z}^2$-SFTs such that there are two $\mathbb{Z}^2$-SFTs X_- and X_+ satisfying the following conditions:

1. X_+ satisfies $\mathscr{P}$.
2. Every $\mathbb{Z}^2$-SFT which factors onto X_- fails to satisfy $\mathscr{P}$.
3. There is a topological morphism from X_+ to X_-.

Then $\mathscr{P}$ is undecidable (from $\mathbb{Z}^2$-SFT presentations).

We called a property as in the statement a **Berger property**. Examples of Berger properties are being minimal, transitive, and having topologically complete topological entropy ([1]). Despite we can prove the undecidability of many dynamical properties with these results, the frontier between decidability and undecidability seems rather nontrivial.

A result for dynamical invariants

Using the same proof idea, we found a generalization of the well-known result that topological entropy of $\mathbb{Z}^2$-SFTs can not be computed from their presentations. Let $\boldsymbol{I}$ be a dynamical invariant taking values in $\mathbb{R}$, which is nonincreasing by factor maps, and for which there are two nonempty SFTs $X \subset X'$ with $\boldsymbol{I}(X) < \boldsymbol{I}(X')$.

Theorem ([1])

There exists no algorithm which on input the presentation of a nonempty $\mathbb{Z}^2$-SFT X and a rational number $\varepsilon > 0$, outputs a rational number whose distance to $\boldsymbol{I}(X)$ is at most ε.

The emptiness problem for SFTs

The main ingredient in the proofs of these results is the fact that in $\mathbb{Z}^2$, the emptiness problem for SFTs is undecidable (Berger's theorem). This result has been extended from $\mathbb{Z}^2$ to many many groups in recent years, and our results admit the following natural generalization:

Theorem ([1])

These results hold for every finitely generated group where the problem of deciding whether an SFT is empty is undecidable.

Bibliography

Nicanor Carrasco-Vargas.
Undecidability of dynamical properties of sfts and sofic subshifts on $\mathbb{Z}^2$ and other groups.

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