Research statement

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Introduction

In this research statement, I will give a hopefully simple explanation of the research that I have conducted until now. I will also mention some ongoing projects, and finally mention some broad and specific interests for future research. The content of this document is summarized in the following table.

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1 Past research.

During the Phd, my research interests have been motivated by recursion-theoretical aspects of shift spaces and subshifts of finite type, but they have also diverged to other topics. This has resulted in the works [6, 7, 5, 1], which will be explained in this section.

A simple example of subshift if finite type on $\mathbb{Z}^2$ is the set of all tilings of $\mathbb{Z}^2$ that we can obtain from a finite set of square 1x1 tiles with some matching rules.

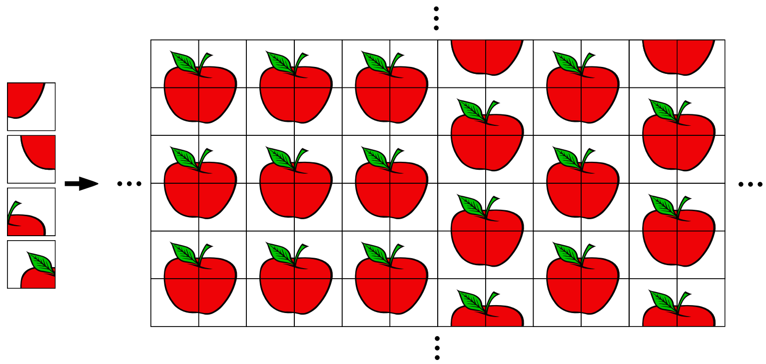

Figure 1: At left, a set of four square tiles with pictures, which convey matching rules. At right, a tiling of a finite portion of the plane, where these matching rules are satisfied.

Factors of subshifts of finite type, and effective dynamical systems

A widely open question in the study of dynamical systems is the following:

Question 1. *Which systems can be topological factors of SFTs?*

In the recent work [1] with Sebastián Barbieri and Cristóbal Rojas, we showed that many natural classes of dynamical systems can be associated to subshifts and SFTs by showing that they are *computable.* Instead of going into details, I will mention an interesting application of our results:

Theorem 2 ([1, Example 5.4])**.** *Let $n \geq 5$. Then the action $GL_n(\mathbb{Z}) \curvearrowright \mathbb{R}^n/\mathbb{Z}^n$ by left matrix multiplication is the topological factor of a $GL_n(\mathbb{Z})$-SFT.*

The reader is referred to the article, where we explore the class of *effective dynamical systems* beyond topological dimension zero.

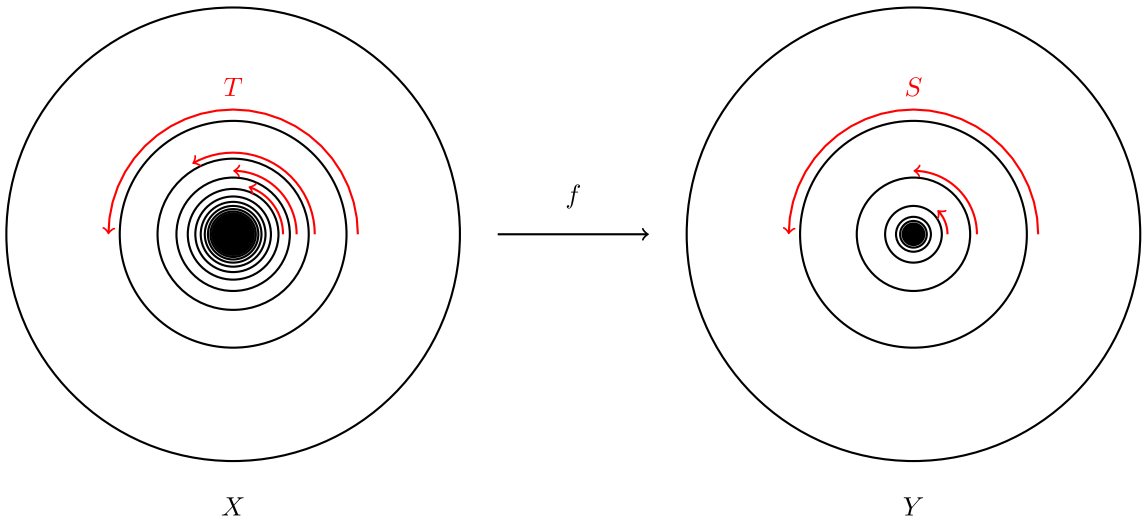

Figure 2: The class of *effective dynamical systems* is not closed by topological factor maps. A counterexample is represented in this picture, see also [1, Figure 1].

Translation-like actions by $\mathbb{Z}$

In the article [29], Whyte proposed to take predicates about groups and subgroups, and replace the subgroup relation by a particular case of group action called *translation-like action.* A group action $G \curvearrowright X$ on a metric space X is called *translation-like* if it is free, and for every g the function $x \mapsto x * g$ is at bounded distance from the identity $X \to X$. He called this a *geometric reformulation.* This idea turned out to be quite useful in the study of shift spaces associated to groups, as translation-like actions can be described by subshifts [15, 16, 9].

In the article [27], Seward showed that the geometric reformulation of Burnside's problem has a positive answer. That is, every finitely generated infinite group admits a translation-like action by $\mathbb{Z}$. He also proved that this action can be taken to be transitive exactly when the group has one or two ends, and this was obtained from a more general result for graphs with uniformly bounded degree. In the article [6], I generalized these results to locally finite graphs:

Theorem 3 ([6])**.** *Let Γ be a graph which is locally finite, connected, and infinite. Then:*

1. *Γ admits a translation-like action by $\mathbb{Z}$.*
2. *Γ admits a transitive translation-like action by $\mathbb{Z}$ if and only if it has one or two ends.*

The actions constructed satisfy $d(v, v * 1) \leq 3$ for every vertex v. Moreover, the proof of the second item is computable.

This generalization had been asked by Seward in Problem 3.5 [27]. My motivation to prove this result, however, is explained in the next subsubsection.

Medvedev degrees of subshifts

The problem of filling the plane with a given set of square 1x1 tiles can be very complex from the algorithmic point of view. Indeed, the following is a classical result:

Theorem 4 ([12, 24]). *There is a finite set of square tiles with pictures, such that every valid tiling* $x\colon \mathbb{Z}^2 \to A$ *is a non-computable function.*

This result can be alternatively stated as follows: there is a nonempty $\mathbb{Z}^2$-SFT whose *Medvedev degree* is non zero. Medvedev degrees are a measure of the algorithmic complexity of a subset of the Cantor set. The class of Medvedev degrees of subshifts of finite type and of effective subshifts has been completely classified for the groups $\mathbb{Z}^d$, $d \geq 1$ [23, 28]. But what happens when we replace $\mathbb{Z}^d$ by some other group? This question has been a source of motivation during my Phd. In particular, I proved the following result, which generalizes a result of Miller [23] for subshifts on $\mathbb{Z}$:

Theorem 5 ([6]). *Let* G *be a group which is infinite, finitely generated, and has decidable word problem. Then the class of Medvedev degrees of effective subshifts on* G *is the class of* Π_1^0 *degrees.*

A natural intuition is the following: if effective subshifts can be complicated in $\mathbb{Z}$, then of course they can be complicated in a group as in the statement. This intuition can be turned into a proof of Theorem 5 only when the group G has an infinite cyclic subgroup, and moreover this subgroup has *decidable subgroup membership problem.* However, it is well known that there are finitely generated infinite groups with no infinite cyclic subgroup.

The proof of Theorem 5 is as follows. We define a geometric reformulation of the decidable subgroup membership problem for translation-like actions, and we prove that every infinite and finitely generated group with decidable word problem admits a translation-like action by $\mathbb{Z}$ with this property. This allows us to construct an effective G-subshift which describes an effective $\mathbb{Z}$-subshift, and has the same Medvedev degree [6].

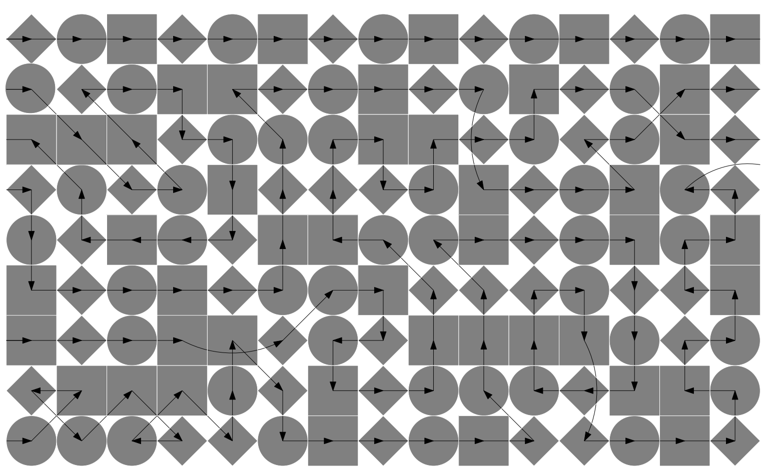

Figure 3: A representation of a $\mathbb{Z}^2$-subshift which contains the information of both a $\mathbb{Z}$-subshift, and a translation-like action by $\mathbb{Z}$ on $\mathbb{Z}^2$.

Are all dynamical properties of $\mathbb{Z}^2$-SFTs undecidable?

A remarkable feature about two-dimensional SFTs is that they can behave as models of computation. This is very interesting, and has as consequence a variety of results linking dynamical properties of $\mathbb{Z}^2$-SFTs with recursion-theoretical ones [14, 22, 13, 2, 17]. On the other hand, a negative consequence is an obstruction to perform simulations or computations on $\mathbb{Z}^2$-SFTs. Indeed, the most basic question one may ask is undecidable:

Theorem 6 (Berger, [4]). *There exists no algorithm which given a $\mathbb{Z}^2$-SFT presentation (a finite alphabet A and a finite set of local rules) decides whether the corresponding SFT is empty or not.*

It is known that many dynamical properties of $\mathbb{Z}^2$-SFTs are undecidable. For this reason, Lind coined the term "swamp of undecidability" [20]. In the article [5] I studied the swamp of undecidability with techniques from recursion theory. I found the following result:

Theorem 7 ([5]). *Every nontrivial dynamical property of sofic $\mathbb{Z}^2$-subshifts is undecidable.*

The class of sofic subshifts is larger than the class of SFTs, and indeed the result above can not be extended to $\mathbb{Z}^2$-SFTs. For instance, the property of having a fixed point is decidable from SFT presentations. However, I proved a general undecidability result which applies to most of dynamically interesting properties (minimal, transitive, zero topological entropy, aperiodic, TCPE, and others). These results generalize from $\mathbb{Z}^2$ to to all groups where the problem of deciding whether an SFT is empty from its presentation is undecidable. The reader is referred to [5] for details.

Infinite Eulerian paths are computable on graphs with vertices of infinite degree

Computing infinite objects associated to infinite graphs is a highly nontrivial problem, as Theorem 4 shows. There is a variety of classical constructions on infinite graphs which are formally impossible to compute, even assuming that the graph is locally finite. This is the case for infinite paths on trees [18], k-colorings [26], perfect matchings [21], Hamiltonian paths [3], and domatic partitions [19]. In slightly informal words, I proved the following:

Theorem 8 ([7, Theorem A]). *Infinite Eulerian paths can always be computed on a graph which admit such paths, as long as one has access to the relation of adjacency, and to the vertex degree function. Vertices of infinite degree are allowed!.*

In fact, for such graphs I proved the following stronger result: in the set of all infinite Eulerian paths, the set of computable ones is dense (see [7, Theorem B]). The interesting thing about these results is that the graph is allowed to have vertices with infinite degree. Previously, it was known for locally finite graphs [3] that there was *at least one* computable infinite Eulerian path.

Remark 9. Infinite Eulerian trails are a key ingredient of the proof of Seward's result on translation-like actions by $\mathbb{Z}$ [27]. This is the reason why I was taken to this topic. The alternative proof of Seward's result that I found, however, does not use infinite Eulerian trails.

2 Current research

Automorphism groups of subshifts

With Sebastián Barbieri and Paola Rivera, we are currently studying automorphism groups of subshifts.

The space of all subshifts

In the recent article [25], Pavlov and Schmieding studied the space of all subshifts over $\mathbb{Z}$, endowed with a natural metric topology (see also [11]). Among other things, they observed that isolated points are dense in this space, and thus one can prove the genericity of different dynamical properties for $\mathbb{Z}$-subshifts by showing that they hold for isolated points.

During the period 2022-2023 I did a research stay with Mathieu Sablik, a the Institut de Mathématiques de Toulouse (France). Among other things we proved the following result, which is in the process of being written:

Theorem 10 (Carrasco-Vargas, Nuñez, Sablik)**.** *Let X be a nonempty $\mathbb{Z}^2$-SFT with no computable elements. Then there is a neighborhood of X in the space of all $\mathbb{Z}^2$-subshifts with no isolated point.*

Of course, this result shows that isolated points are not dense anymore on the space of subshifts on $\mathbb{Z}^2$. This result can be extended from $\mathbb{Z}^2$ to any recursively presented group, and indeed motivates the question of which groups admit nonempty SFTs with no computable element. In other words, SFTs with non zero Medvedev degree.

Medvedev degrees of SFTs

With Sebastián Barbieri, we are currently working on the following question:

Question 11. *Let G be a finitely generated group. Which is the class of Medvedev degrees of subshifts of finite type on G?*

Computability aspects of ends of graphs

The proofs of Theorem 3 and Theorem 8 require a rather good understanding of the connected components that remain in a graph after we remove a finite portion, and in particular, computability aspects of the concept of *ends*. This lead to an ongoing project with Valentino Delle Rose and Cristóbal Rojas.

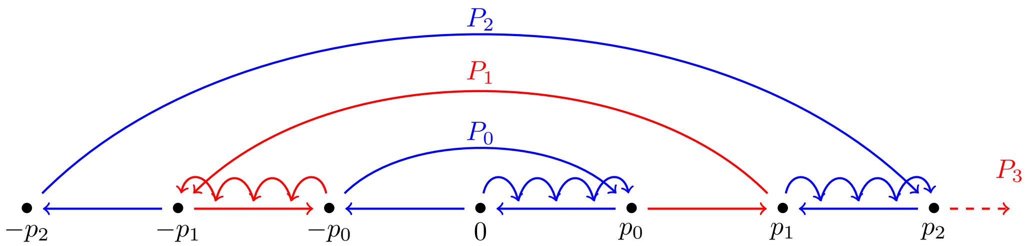

Figure 4: How hard is deciding whether an infinite graph is Eulerian? We showed that in some cases, the complexity actually comes from the problem of detecting the number of ends of the graph. This picture was made by Valentino Delle Rose.

3 Future research

I consider myself rather open to different mathematical directions. I am particularly interested in computability theory, geometric aspects of group theory, and dynamical systems.

Now, I will mention some more specific topics in which I am interested. The first of them is about the existence of aperiodic SFTs.

Question 12. *Which is the class of finitely generated groups admitting weakly aperiodic (resp. strongly aperiodic) subshifts of finite type?*

Let me recall that in a way that can be made precise, an aperiodic SFT is the same as an aperiodic tiling. There are conjectures about the weakly aperiodic case in [8], and the strongly aperiodic case in [10]. Despite there are several works on this topic, the question seems far from being settled.

More broadly, I am also interested in shift spaces over different groups, and the interaction of these spaces with the geometry of the group.

Question 13. *What properties of groups are captured by their shift spaces?*

Indeed, a classical topic in group theory is understanding groups through their group actions. This program has been carried away from different perspectives. Due to my formation, I am particularly interested in what can be said from the algorithmic point of view:

Question 14. *How can we relate algorithmic properties of finitely generated groups, with computability properties of their group actions?*

I will illustrate this vague idea with a result proved by Emannuel Jeandel: if a recursively presented group admits a strongly aperiodic SFT, then it has decidable word problem [15].

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