Automorphism groups of subshifts

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The Curtis-Hedlund-Lyndon theorem

Consider the shift topological dynamical system $(A^{\mathbb{Z}}, \sigma)$, where A is finite and $|A| \geq 2$.

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A function $\phi\colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is continuous and commutes with the shift

$$\phi \circ \sigma = \sigma \circ \phi$$

if and only if ϕ is a cellular automaton.

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This characterizes the **endomorphisms** of the system $(A^{\mathbb{Z}}, \sigma)$.

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Example (Rule 122)

Let $A = \{0,1\}$, consider the local rule $R\colon \{0,1\}^{\{-1,0,1\}} \to \{0,1\}$ by

$$111 \to 0 \; 011 \to 1 \; 101 \to 1 \; 001 \to 1$$

$$110 \to 1 \; 010 \to 0 \; 100 \to 1 \; 000 \to 0$$

This defines a map $\phi\colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ by

$$x \mapsto \phi(x), \; \phi(x)_n = R(x_{n-1}x_nx_{n+1})$$

Nice simulator in https://elife-asu.github.io/wss-modules/modules/1-1d-cellular-automata/

Definition (Cellular atomaton)

A function $\phi\colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a **cellular atomaton** if there is a number $\ell \in \mathbb{N}$ and a rule

$$R\colon A^{\{-\ell,\dots,\ell\}} \to A$$

such that for all $x \in A^{\mathbb{Z}}$ its image $\phi(x)$ is given by

$$\phi(x)_n = R(x_{n-\ell} \dots x_n \dots x_{n+\ell})$$

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The Curtis Hedlund Lyndon theorem rephrased

Endomorphisms of $(A^{\mathbb{Z}}, \sigma)$ = all cellular automata over $A^{\mathbb{Z}}$.
Automorphisms of $(A^{\mathbb{Z}}, \sigma)$ = invertible cellular automata over $A^{\mathbb{Z}}$.

Automorphisms form a group

Observation

If f, g are automorphisms of (X, T), then the function composition

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Definition

The automorphism group $\mathrm{Aut}(X, T)$ (or $\mathrm{Aut}(X)$) is the group whose elements are automorphisms of (X, T), with the group operation of composition.

The rest of the talk is about the automorphism group $\mathtt{Aut}(X)$ of a subshift X.

This subject began with a paper of Hedlund (1969).

Many things are known, some important questions are still open.

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(See Hedlund 1969, Boyle and Krieger 1987, Salo 2018, ...)

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A number of works prove that the slower the complexity function of a subshift, the “smaller” the automorphism group gets in an algebraic sense.
See Cyr and Kra 2015, 2016, 2020, Cyr and Franks and Kra 2019, Cyr and Franks and Kra and Petite 2018, Pavlov and Schmieding 2022...

An open question

The following old quesiton is open

Is $\mathrm{Aut}(\{0,1\}^{\mathbb{Z}})$ *isomorphic to* $\mathrm{Aut}(\{0,1,2\}^{\mathbb{Z}})$*?*

Partial information is given by a result of Kim and Roush, and a result of Ryan.

The theorem of Kim and Roush

Theorem (Kim and Roush 1990)

$\mathrm{Aut}(A^{\mathbb{Z}})$ *embeds into* $\mathrm{Aut}(X)$ *for any mixing* $\mathbb{Z}$*-SFT* X, $|A| \geq 2$.

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Corollary

$\mathrm{Aut}(\{0,1\}^{\mathbb{Z}})$ and $\mathrm{Aut}(\{0,1,2\}^{\mathbb{Z}})$ can not be distinguished by their subgorups.

The theorem of Ryan

If G is a group, its center is

$$Z(G) = \{g \in G : \forall h \in G,\ gh = hg\}$$

Theorem (Ryan, 1972)

$$Z(\mathrm{Aut}(A^{\mathbb{Z}})) = \langle \sigma^n : n \in \mathbb{Z} \rangle, \quad |A| \geq 2.$$

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It can be used to prove:

Theorem (Folklore)

$\mathrm{Aut}(\{0,1\}^{\mathbb{Z}})$ *and* $\mathrm{Aut}(\{0,1,2,3\}^{\mathbb{Z}})$ *are nonisomorphic.*

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In the case of $\{0,1\}^{\mathbb{Z}}$ and $\{0,1,2\}^{\mathbb{Z}}$, the same argument only shows that in both groups the shift has no roots.

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For $\mathbb{Z}$-subshifts our results are new and impliy those of Kim and Roush and Ryan.

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A subshift $X \subset A^G$ is **strongly irreducible** if there is a finite set $K \subset G$ with the following property. If $F, T \subset G$ are finite sets and $FK \cap T = \emptyset$, then for every pair of "words" $p \colon F \to A$ and $q \colon T \to A$ in the language of X, there exist $x \in X$ with $x|_F = p$ and $x|_T = q$.

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Examples

The fullshift A^G, subshifts with a safe symbol, the singleton subshift.

Ward's theorem

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Theorem (Barbieri, Carrasco-Vargas, Rivera-Burgos)

Let G be a countably infinite group, and let X be a strongly irreducible subshift with $|X| \neq 1$. Then $\mathrm{Aut}(X)$ contains copies of every finite group.

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Corollary

All the many groups that are known to embed into $\mathrm{Aut}(A^{\mathbb{Z}})$ also embed into $\mathrm{Aut}(A^G)$.

Generalization 2 of the theorem of Kim and Roush

Theorem (Barbieri, Carrasco-Vargas, Rivera-Burgos)

Let G be a countably infinite group that contains a copy of the free group $\mathbb{F}_k$ on $k \geq 1$ elements, and let X be a strongly irreducible subshift with $|X| \neq 1$.
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For $G = \mathbb{Z}^2$ it is not known whether the automorphism groups of different fullshifts embed into each other.

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Corollary

The automorphism groups of $\{0,1\}^{\mathbb{Z}^n}$ and $\{0,1\}^{\mathbb{Z}^m}$ are nonisomorphic for $n \neq m$.

Application of Ryan's theorem

Theorem (Barbieri, Carrasco-Vargas, Rivera-Burgos)

Let G be a countable group of the form $H \times \mathbb{Z}$. Let A, B such that $|A|$ and $|B|$ are different powers some natural number $\ell \geq 2$. Then $\mathrm{Aut}(A^G)$ *and* $\mathrm{Aut}(B^G)$ *are nonisomorphic.*

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Proof idea

We can choose a subshift $Y \subset A^G$ and $g \in Z(G)$ sucht that $\sigma^g \curvearrowright Y$ is conjugate to a mixing $\mathbb{Z}$-SFT, and use an entropy argument.

Thanks

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