# Slow entropy of some skew products

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https://arxiv.org/abs/2506.17932

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In this talk all systems have discrete time $\mathbb{Z}$ or $\mathbb{N}$ (no flows)

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Often they will be invertible, but not always

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3. Natural problem: how can we distinguish $S \rtimes_\tau T_1$ and $S \rtimes_\tau T_2$?
4. This problem is well-studied for measure preserving systems; we study it for topological systems.
5. We propose a solution based on slow entropy, an invariant introduced by Katok and Thouvenot.

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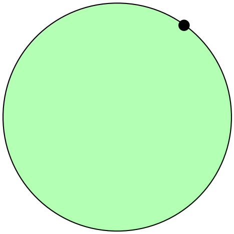

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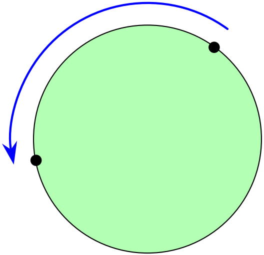

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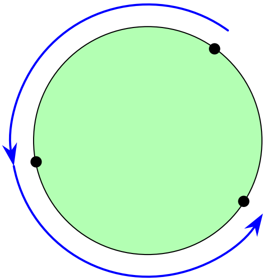

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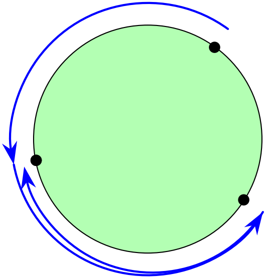

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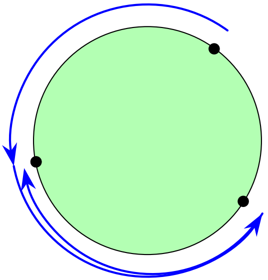

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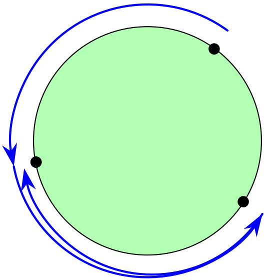

Take the space

$$\{-1, 1\}^{\mathbb{Z}} \times \mathbb{T}$$

and the transformation

$$(y, x) \mapsto (\sigma(y), T^{y(0)}(x))$$

# The $[T, T^{-1}]$ system

The $[T, T^{-1}]$ system is the skew product

- Base: $\sigma \curvearrowright (\{-1,1\}^{\mathbb{Z}}, \mu_{\frac{1}{2},\frac{1}{2}})$
- Fiber: $T \curvearrowright (X, \mathcal{X}, \mu)$

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In 1982 Kalikow proved that if $T$ has positive entropy, then $\sigma \rtimes T$ is $K$ and not Bernoulli (first "natural" known examples).

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One can prove that all of these systems have the same entropy

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**Theorem (Austin 2015)**

*Yes.*

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The new transformation is

$$S \rtimes_\tau T \curvearrowright (Y, \mathcal{Y}, \nu) \times (X, \mathcal{X}, \mu)$$

$$(y, x) \mapsto (S(y), T^{\tau(y)}(x))$$

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3. A mixing SFT $S \curvearrowright (Y, \mathcal{Y}, \nu)$, $\nu$ a Gibbs measure, and $\tau \colon Y \to \mathbb{Z}$ satisfying a technical condition (Austin 2015)

# What happens in the general case?

For which choices of $S \curvearrowright (Y, \mathcal{Y}, \nu)$ and $\tau\colon Y \to \mathbb{Z}$ is it true that the fiber entropy is an isomorphism invariant in this corresponding class of skew products?

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- What happens for topological systems?
- We propose a solution, it works for a diverse class of $S$ and $\tau$.

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## The class of systems

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## Specific problem

Determine if ~~entropy~~ *entropy type* invariants of the fiber $T$ are an invariant for topological conjugacy within this class.

# Intuition

The problem is interesting when the entropy of the fiber "disappears". Informally,

$$\# \text{ of } (n, \epsilon)\text{-Bowen balls} \approx e^{n \cdot h_{top}(S) + f(n) h_{top}(T)}$$
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for some sublinear $f(n)$.
We want to capture the contribution of $h_{top}(T)$ in this relation.

$$\# \text{ of } (n, \epsilon)\text{-Bowen balls} := \mathrm{spa}(T, n, \epsilon) \quad n \in \mathbb{N}, \epsilon > 0$$
needed to cover the space

# Topological slow entropy

## Definition (Katok and Thouvenot '97)

Let $\mathbf{a} = \{a_n(t)\}_{n\in\mathbb{N},t>0}$ be a family of sequences with

- for each $n$, $t \to a_n(t)$ is monotone,
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The upper slow entropy of $(X, T)$ with scale $\mathbf{a}$ is

$$\overline{\text{ent}}_{\mathbf{a}}(T) = \lim_{\epsilon\to 0} \overline{\text{ent}}_{\mathbf{a}}(T, \epsilon)$$

$$\overline{\text{ent}}_{\mathbf{a}}(T, \epsilon) = \sup(\{0\} \cup \{t > 0 : \limsup_{n\to\infty} \frac{\text{spa}(T, n, \epsilon)}{a_n(t)} > 0\})$$

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- With scale $\{n^t\}_{n\in\mathbb{N},t>0}$ we obtain the so called polynomial complexity or polynomial entropy of the system.

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**Theorem (C, arXiv:2506.17932)**

*Let* $(Y, S)$ *be a subshift and let* $\tau\colon Y \to \mathbb{Z}$ *be continuous. Then there is a scale* $\boldsymbol{a}$ *depending only on* $(Y, S, \tau)$ *such that for every invertible topological dynamical system* $(X, T)$ *we have*

$$\overline{\mathrm{ent}}_{\boldsymbol{a}}(S \rtimes_\tau T) = \underline{\mathrm{ent}}_{\boldsymbol{a}}(S \rtimes_\tau T) = h_{top}(T),$$

*provided that* $\tau$ *satisfies the condition of being* $\lambda$*-unbounded for some* $\lambda > 0$ *(defined soon).*

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*provided that $\tau$ satisfies the condition of being $\lambda$-unbounded for some $\lambda > 0$ (defined soon).*

This shows that the entropy of the fiber is a conjugacy invariant in the corresponding class of skew products $S \rtimes_\tau T$.

# A relative version of the main result

**Theorem (C, arXiv:2506.17932)**

*Let* $(Y, S)$ *be a subshift and let* $\tau\colon Y \to \{-1, 0, 1\}$ *be continuous. For every scale* $\mathbf{b}$ *we can find a scale* $\mathbf{c}$ *such that for every invertible topological dynamical system* $(X, T)$ *we have*

$$\underline{\mathrm{ent}}_{\boldsymbol{b}}(T) \leq \overline{\mathrm{ent}}_{\boldsymbol{c}}(S \rtimes_\tau T) \leq \overline{\mathrm{ent}}_{\boldsymbol{b}}(T)$$

*provided that* $\tau$ *satisfies the condition of being* $\lambda$*-unbounded for some* $\lambda > 0$ *(defined soon).*

This shows that for any scale **b**, the *slow entropy* with scale **b** of the fiber is a conjugacy invariant in this class of skew products, but we must restrict to fiber systems with the property $\overline{\mathrm{ent}}_{\mathbf{b}}(T) = \underline{\mathrm{ent}}_{\mathbf{b}}(T)$.

# The property of being $\lambda$-unbounded

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- For $y \in Y$ we interpret $N \to \sum_{n=0}^{N-1} \tau(S^n(y))$ as a walk on $\mathbb{Z}$, and we define the *range* $R_N(y)$ as the set of places in $\mathbb{Z}$ it visits in $N$ steps:

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- $R_N(y)$ depends only on $y_0 \dots y_{N-1}$, so we can also write $R_N(w)$ for a word $w$ of length $N$ that appears in $Y$.

## Definition

Given a subshift $(Y, S)$ and a cocycle $\tau\colon Y \to \mathbb{Z}$ depending only on the zero coordinate of the configuration, we say that $\tau$ is $\lambda$-unbounded ($\lambda > 0$) if for all $C \in \mathbb{N}$

$$\liminf_{n\to\infty} \frac{|\{w \in L_n(Y) : |R_n(w)| \geq C\}|}{|L_n(Y)|} \geq \lambda$$

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- If $R_n(y)$ unbounded for all $y$, then this holds.
- Also true for $(\{-1, 1\}^{\mathbb{Z}}, \mu_{\frac{1}{2},\frac{1}{2}})$ and $\tau(y) = y(0)$.

# Examples

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**Example**

$(Y, S)$ is a minimal subshift and $\tau\colon Y \to \mathbb{Z}$ is not a coboundary (i.e. $\tau = g - g \circ S$ for some continuous $g\colon Y \to \mathbb{R}$).

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- For any $T$ we have $h_{top}(S \rtimes_\tau T) = h_{top}(S) + \alpha h_{top}(T)$, so applying the main theorem is an overkill.

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$Y$ is a Sturmian subshift on symbols $\{0, 1\}$ and $\tau(y) = y(0)$.

- For any $T$ we have $h_{top}(S \rtimes_\tau T) = h_{top}(S) + \alpha h_{top}(T)$, so applying the main theorem is an overkill.
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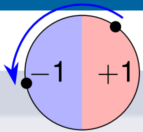
The deterministic random walk (the subshift copy).




## Example

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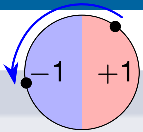
- Choose an irrational $\alpha$, and take the function $\rho\colon [0,1) \to \{-1,1\}$ with value 1 on $[0,1/2)$ and value $-1$ over $[1/2,1)$.

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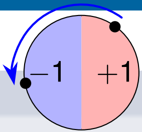


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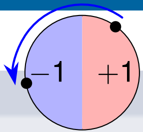


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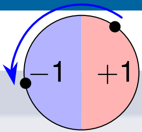


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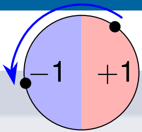


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and

$$\mathrm{spa}(T, F, \epsilon)$$

equals the minimal number of Bowen $d_F, \epsilon$-balls needed to cover $X$.

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We can ignore the shape of $R_N(w)$ and only care about their cardinality. Remember that $\mathrm{spa}(T, \{0, \ldots, N-1\}, \epsilon) \approx e^{h_{top}(T)N}$.

$$\approx \sum_{w \in L_n(Y)} e^{|R_N(w)| h_{top}(T)}$$

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We have $\overline{\mathrm{ent}}_{\mathbf{a}}(S \rtimes_\tau T) = h_{top}(T)$ because

- $t < h_{top}(T)$ implies $\sum_{w \in L_n(Y)} e^{|R_N(w)| t} << \sum_{w \in L_n(Y)} e^{|R_N(w)| h_{top}(T)}$
- $t > h_{top}(T)$ implies $\sum_{w \in L_n(Y)} e^{|R_N(w)| t} >> \sum_{w \in L_n(Y)} e^{|R_N(w)| h_{top}(T)}$

# General case

If $\tau$ takes values outside $\{-1, 0, 1\}$, the sets $R_N(w)$ may have holes, and this causes extra growth of $\text{spa}(T, R_N(w), \epsilon)$.
For instance, if $R_N(w) = \{2, 4, \ldots, 2N\}$,

$$\text{spa}(T, R_N(w), \epsilon) \approx e^{\mathbf{2} \cdot h_{top}(T)N}$$

It is possible to quantify this extra contribution and include it to the scale, so that they “cancels out”.

# Comments and questions

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The scales constructed here don’t work for these purposes (in general).

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