A dynamical invariant for subshifts of recursive nature

Nicanor Carrasco-Vargas

UC, Chile
njcarrasco@mat.uc.cl
Beamer in nicanorcarrascovargas.github.io

CIRM 2024
Some of these results are part of my Phd thesis under the advise of Cristóbal Rojas (UC) and Sebastián Barbieri (USACH), and in colaboration with Alonso H. Núñez (IMT) and Mathieu Sablik (IMT).

Contents

Topic of this talk: a dynamical invariant for subshifts, m.

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Preliminaries

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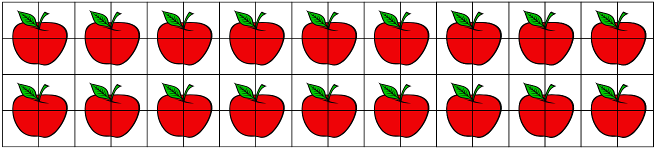

Shift spaces in the real world

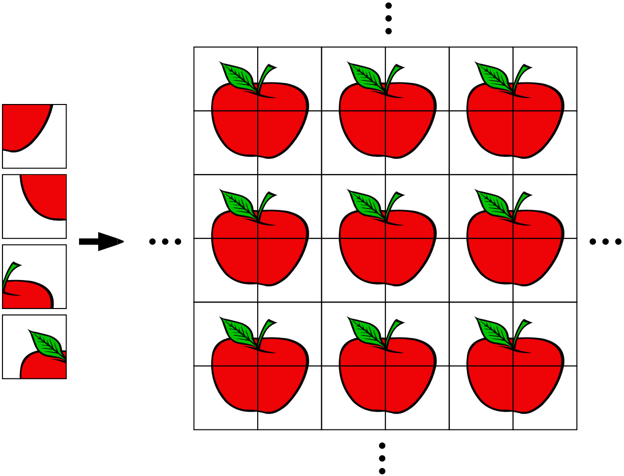

Figure: Four tiles with pictures, from which we obtain a set of infinite tilings of $\mathbb{Z}^2$.

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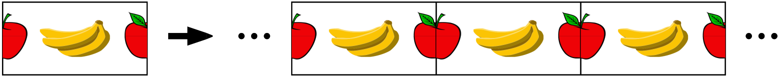


Figure: One tile with pictures, from which we obtain a set of infinite tilings of $\mathbb{Z}$.

Shift spaces on $\mathbb{Z}^d$

1. Alphabet $A =$ a finite set (of colors, tiles, symbols, etc).
2. Configuration $= x\colon \mathbb{Z}^d \to A$.
3. Subshift $\mathbb{Z}^d =$ topologically closed subset of $\{x\colon \mathbb{Z}^d \to A\}$ for some alphabet A, which is invariant under the action $\mathbb{Z}^d \curvearrowright \{x\colon \mathbb{Z}^d \to A\}$ by translations:

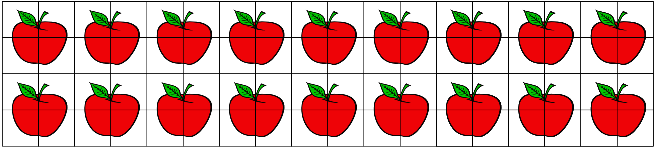
$$x \mapsto \sigma^{\boldsymbol{n}} x$$

$$(\sigma^{\boldsymbol{n}} x)(\boldsymbol{m}) = x(\boldsymbol{m} - \boldsymbol{n}).$$

4. These definitions can be easily extended to a countable group.

SFT = subshift of finite type

Up to topological conjugacy, an SFT on $\mathbb{Z}^d$ is the same as a space of tilings obtained from tiles with pictures



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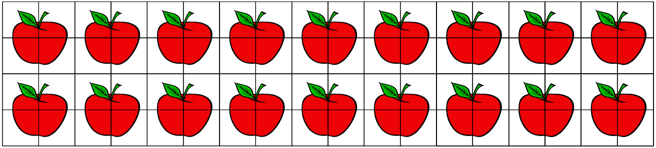
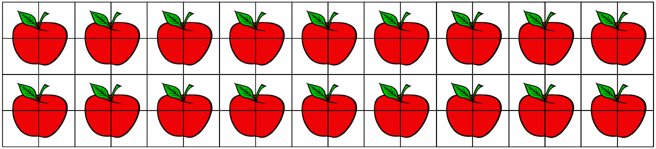


Formally, a subshift is of finite type if it can be defined by a finite set of forbidden patterns ($p : \{0, \dots, n\}^d \to A$).

Periodicity and computability

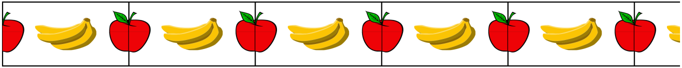
Motivation I: Periodicity and computability



Periodicity and computability

Theorem (Folklore)

In $\mathbb{Z}$, every SFT has a periodic configuration.



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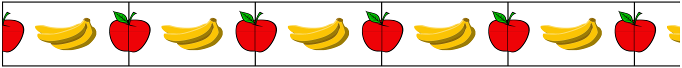

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In $\mathbb{Z}^2$, there exists an SFT with no periodic configurations (no finite orbits).

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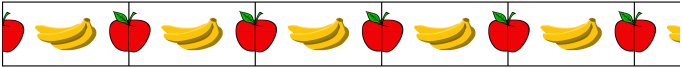

Theorem (Berger 1966)

In $\mathbb{Z}^2$, there exists an SFT with no periodic configurations (no finite orbits).

Theorem (Hanf and Myers, 1974)

In $\mathbb{Z}^2$ there is an uncomputable SFT, that is, all its configurations are uncomputable.

Computability

Definition

A configuration $x : \mathbb{Z}^d \to A$ is computable if there is a computer program which on input $\boldsymbol{n} \in \mathbb{Z}^d$, outputs $x(\boldsymbol{n})$.

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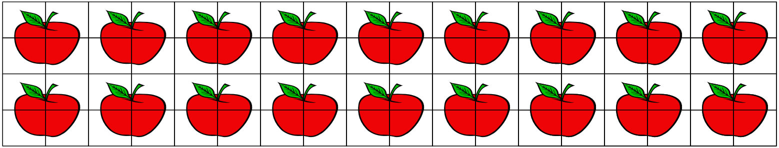
Example

A periodic configuration in $A^{\mathbb{Z}^2}$ (the infinite repetition of a $n \times m$ pattern) is also computable.



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Theorem

For subshifts, uncomputable $\Rightarrow$ aperiodic (no configuration has finite orbit)

These properties are valid for finitely generated groups!

Periodicity and computability

It is an open question which groups:

admit aperiodic SFTs

admit uncomputable SFTs

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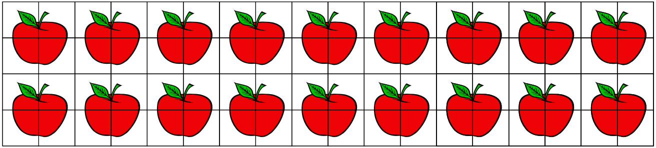
admit aperiodic SFTs

admit uncomputable SFTs

In many groups, the techniques which have allowed to answer one question, have also allowed to answer the other.

Periodicity and computability

Motivation II: The space of all subshifts and its isolated points



The space of all subshifts

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$$S^d = \{X \text{ subshift on } \mathbb{Z}^d \text{ with alphabet and } A \subset \mathbb{N}\}$$

$$d(X, Y) = 2^{-n}, \quad n = \max\{k \mid L_k(X) = L_k(Y)\}$$

$$L_k(X) = \{x|_{B_k} : x \in X\}, \ B_k = \{-k, \dots, k\}^d \subset \mathbb{Z}^d.$$

The space S^1

Theorem (Pavlov and Schmieding 2022)

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It follows that to study generic properties in S^1, it is **enough** to do it on isolated points.

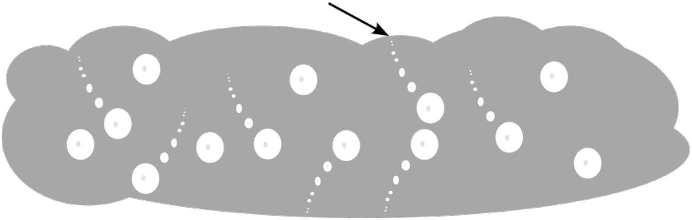

Figure: The space S^1

The space S^2

Theorem (C, Núñez, y Sablik)

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The space S^2

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It follows that to study generic properties in S^2, it is **not enough** to do it on isolated points.

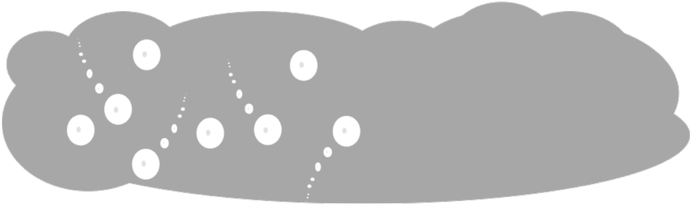

Figure: The space S^2

El espacio S^2

Theorem (C, Núñez, y Sablik)

If X is an uncomputable SFT, then it has a neighborhood with no isolated points.

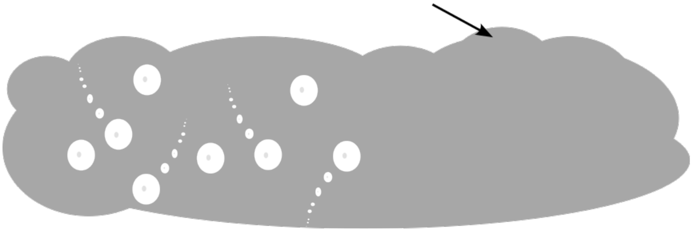

Figure: *The space S^2*

The invariant m

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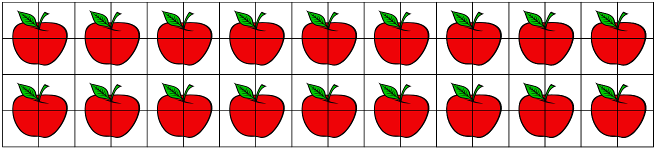

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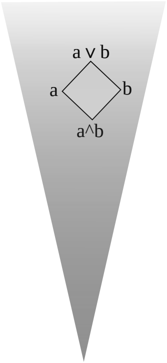
It takes values on a partially ordered set $(M, \leq_M)$ which is a lattice and has a minimal element 0_M.



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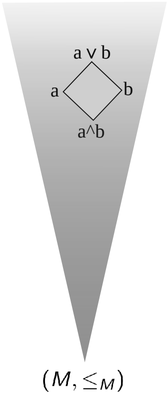

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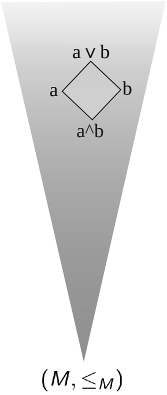

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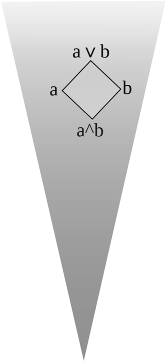

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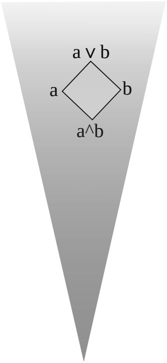

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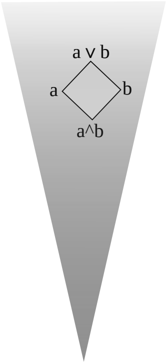

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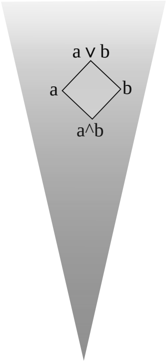

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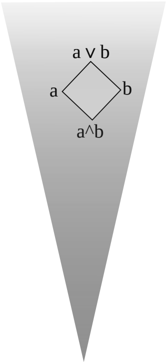
- These properties hold for G finitely generated!
- Compare with topological entropy for amenable groups.

Formal definition



$(M, \leq_M)$

Definition (Medvedev 1955)

Let $P, Q \subset \{0,1\}^{\mathbb{N}}$. We write

$$P \leq_M Q$$

when there is a computable function $Q \to P$.

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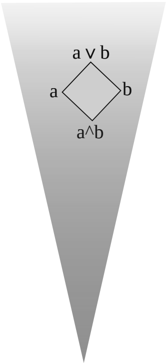

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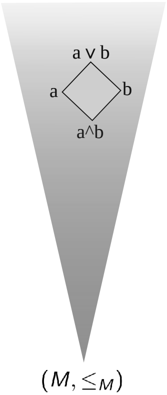

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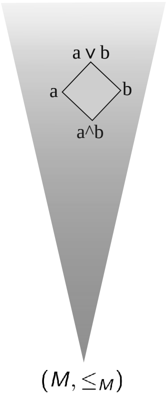

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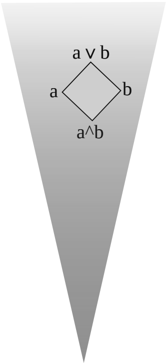

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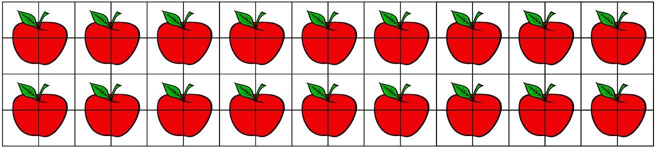
- This extends to subsets of A^G.

What is it known?

What is it known about m as dynamical invariant for sub-shfits?



The classification problem

Classic question:

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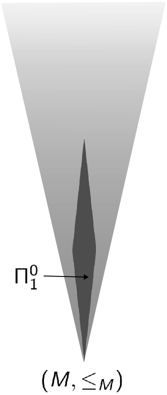
This question has been implicit in the literature for some time. My thesis is about this question.

What is known about SFTs



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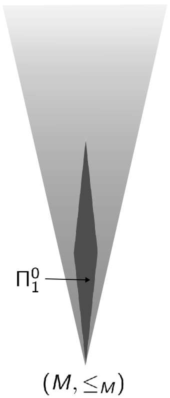

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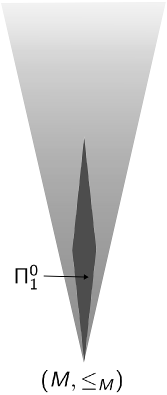

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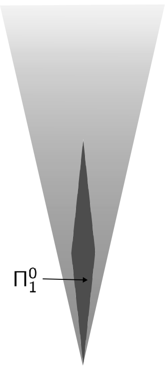
- In $\mathbb{Z}^d$, $d \geq 2$, SFTs attain all Π^0_1 elements in the lattice M. (Simpson, 2012)
- What happens in other groups?
  This problem is rather hard, and we studied the classification problem for effective subshifts.

SFT → effective subshifts



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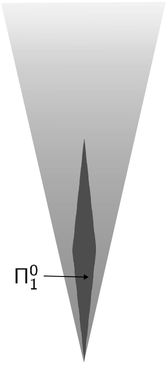
SFT $\to$ effective subshifts



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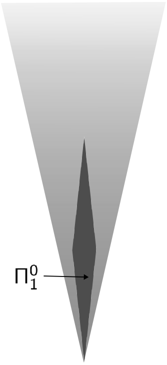

- Effective subshifts = of subshifts which contains SFTs and sofic subshifts
- In $\mathbb{Z}$, effective subshifts attain all Π^0_1 elements in the lattice M. (Miller, 2011)
- This classification extends to every finitely generated group which is infinite and has decidable word problem (The geometric subgroup membership problem, N. C. *Preprint*).

Thanks!

Thanks!

Bibliografia

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