Isomorphisms of skew products over irrational rotations

Seminar talk for Dynamics Seminar at Jagiellonian University, Poland

Nicanor Carrasco-Vargas

Jagiellonian University
nicanor.vargas@uj.edu.pl
Beamer $\in$ nicanorcarrascovargas.github.io

April 17, 2026

Introduction

Disclaimer

Please interrupt me.

INTRODUCTION
DISCLAIMER

Please interrupt me.

In this talk all systems are measure-preserving and invertible.

INTRODUCTION
DISCLAIMER

Please interrupt me.

In this talk all systems are measure-preserving and invertible.

Transformations always act on the probability space

$$(X, \mathcal{B}, \mu) = ([0, 1], \text{ Borel sigma algebra}, \text{Lebesgue measure})$$

or something isomorphic.

Disclaimer

Please interrupt me.

In this talk all systems are measure-preserving and invertible.

Transformations always act on the probability space

$$(X, \mathcal{B}, \mu) = ([0, 1], \text{ Borel sigma algebra}, \text{Lebesgue measure})$$

or something isomorphic.

Measurable = Borel measurable.

Introduction
Disclaimer

Please interrupt me.

In this talk all systems are measure-preserving and invertible.

Transformations always act on the probability space

$$(X, \mathcal{B}, \mu) = ([0,1], \text{ Borel sigma algebra}, \text{Lebesgue measure})$$

or something isomorphic.

Measurable = Borel measurable.

Dynamical system = measure preserving system = transformation = automorphism

INTRODUCTION
DISCLAIMER

Please interrupt me.

In this talk all systems are measure-preserving and invertible.

Transformations always act on the probability space

$$(X, \mathcal{B}, \mu) = ([0, 1], \text{ Borel sigma algebra}, \text{Lebesgue measure})$$

or something isomorphic.

Measurable = Borel measurable.

Dynamical system = measure preserving system = transformation = automorphism

No flows and no group actions (time = $\mathbb{Z}$).

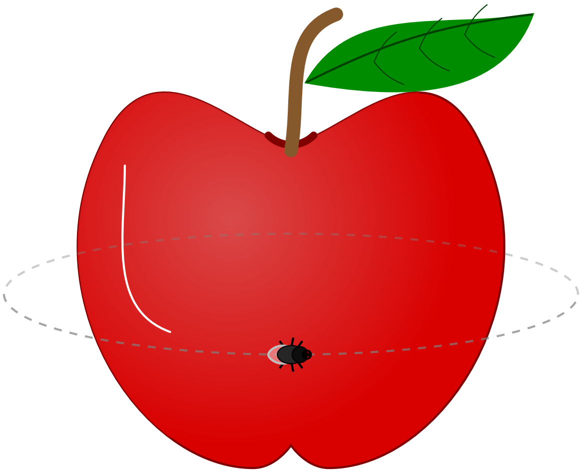
Introduction

Kalikow's $[T, T^{-1}]$ system

The classic $[T, T^{-1}]$ system is a skew product construction which formalizes the idea of applying a transformation T and its inverse T^{-1} randomly.

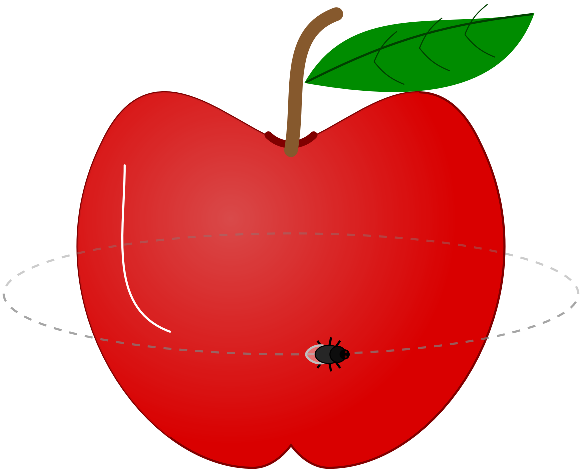
Introduction

Kalikow's $[T, T^{-1}]$ system



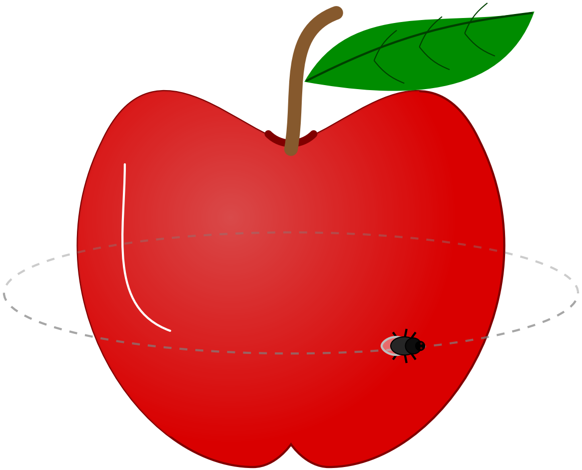
Introduction

Kalikow's $[T, T^{-1}]$ system



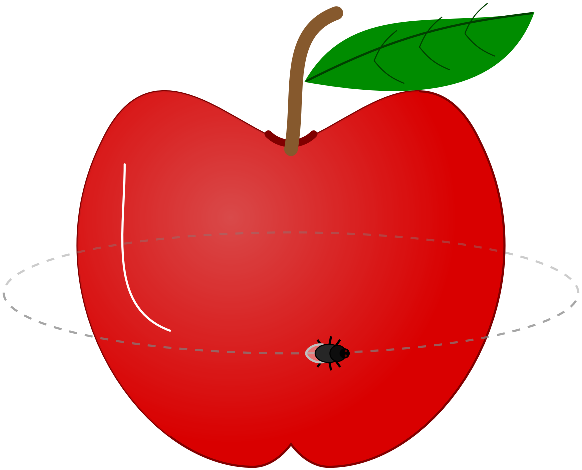
Introduction

Kalikow's $[T, T^{-1}]$ system



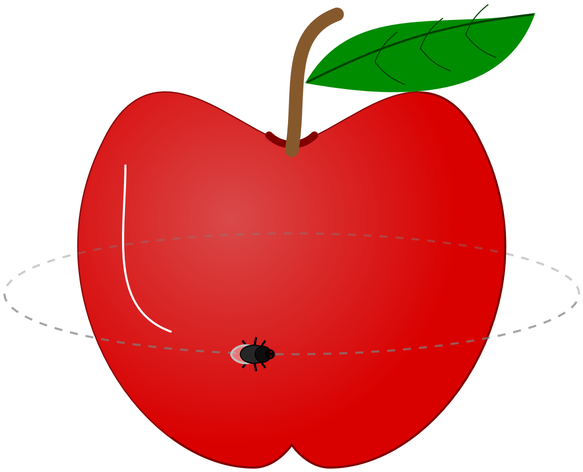
Introduction

Kalikow's $[T, T^{-1}]$ system



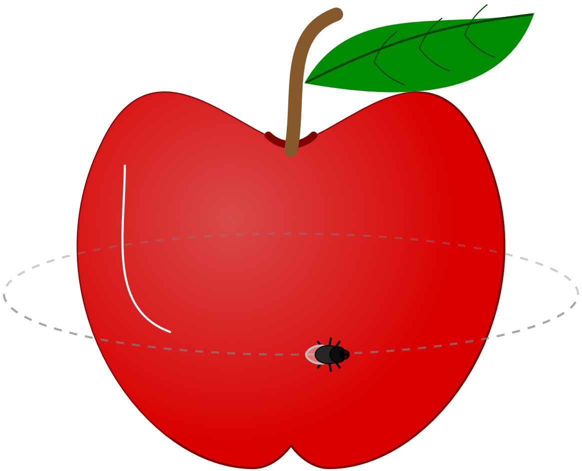
Introduction

Kalikow's $[T, T^{-1}]$ system



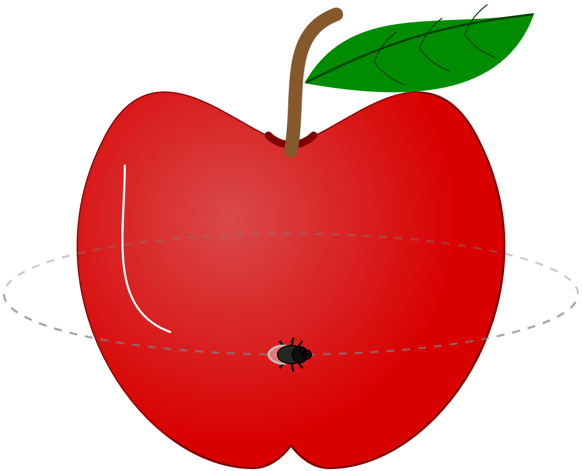
Introduction

Kalikow's $[T, T^{-1}]$ system



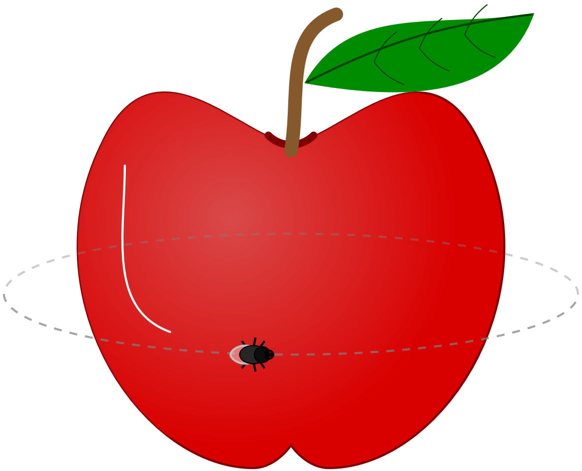
Introduction

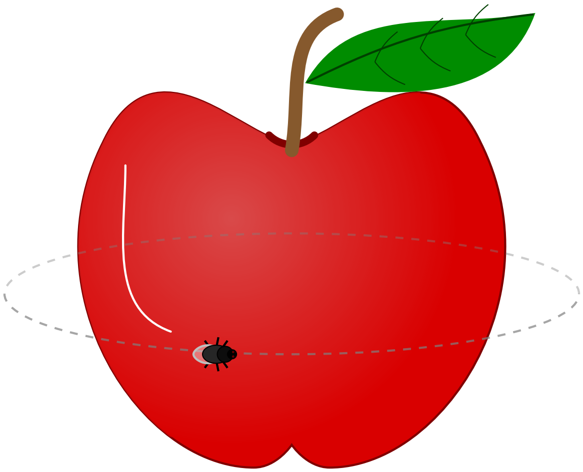
Kalikow's $[T, T^{-1}]$ system



Introduction

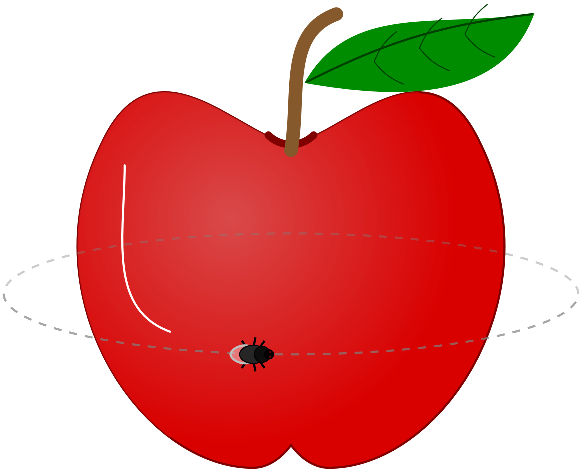
Kalikow's $[T, T^{-1}]$ system

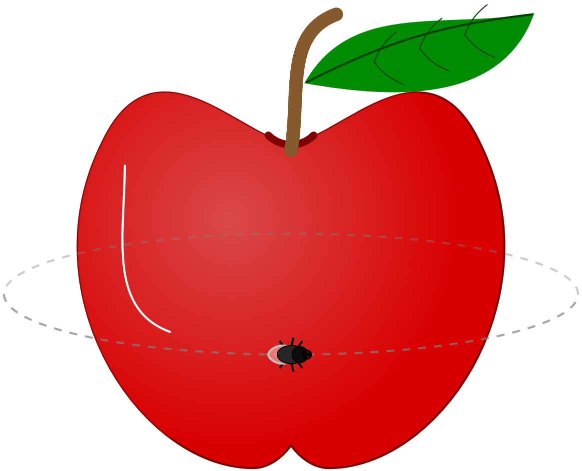




Introduction

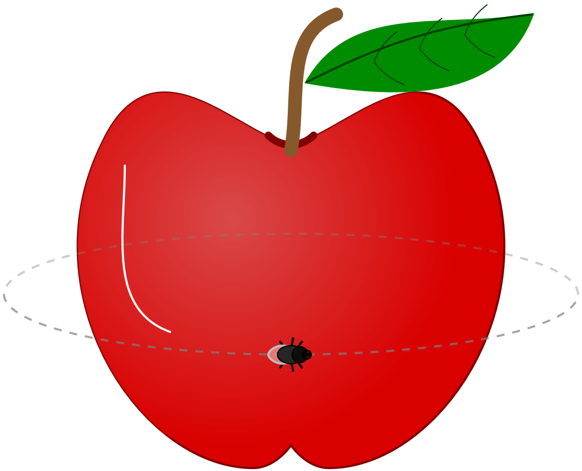
Kalikow's $[T, T^{-1}]$ system

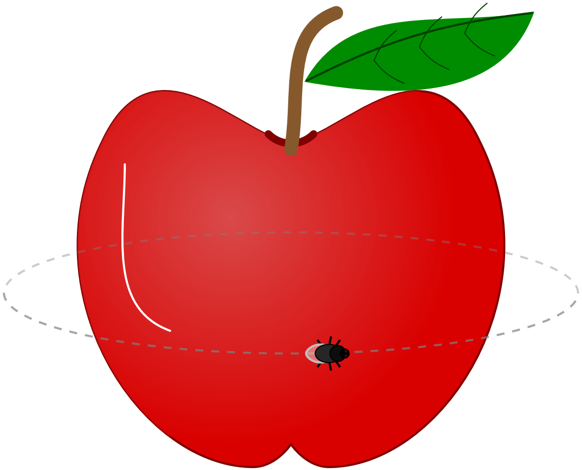




Introduction

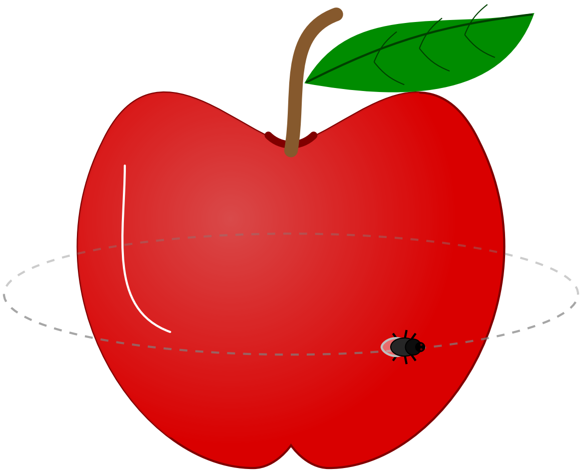
Kalikow's $[T, T^{-1}]$ system





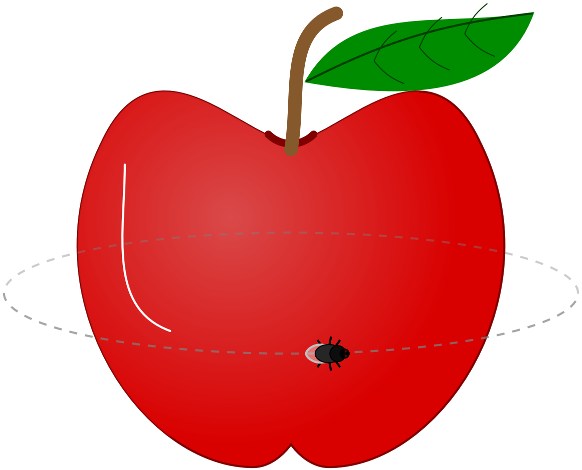
Introduction

Kalikow's $[T, T^{-1}]$ system



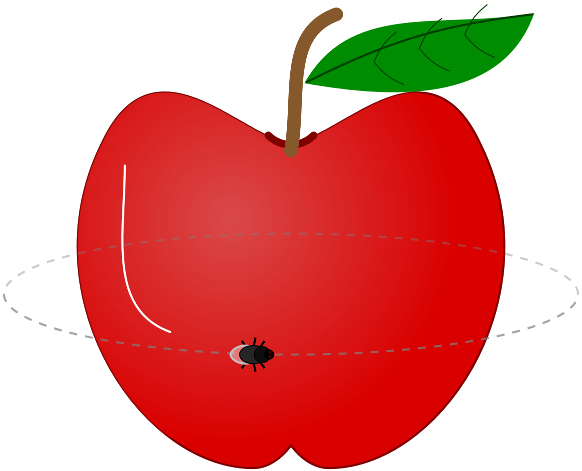
Introduction

Kalikow's $[T, T^{-1}]$ system



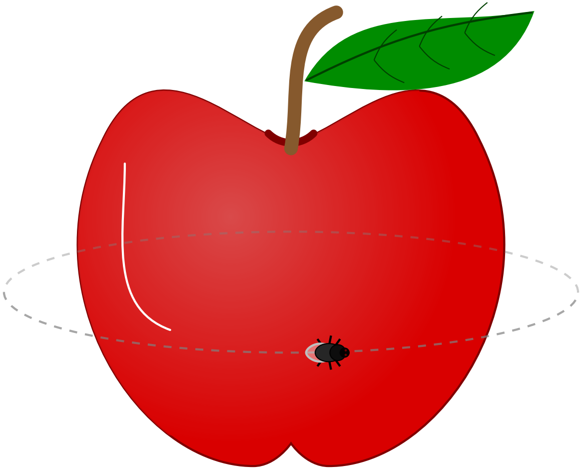
Introduction

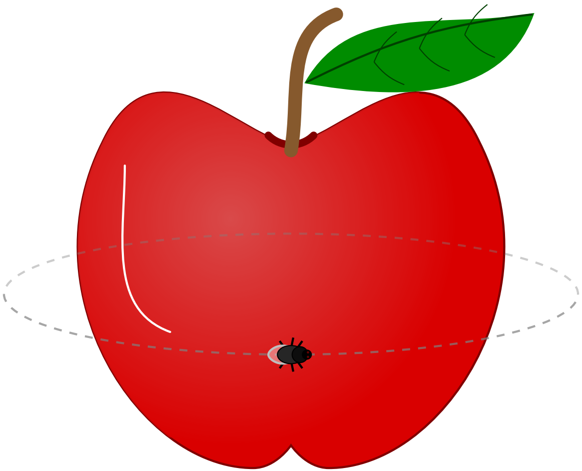
Kalikow's $[T, T^{-1}]$ system



Introduction

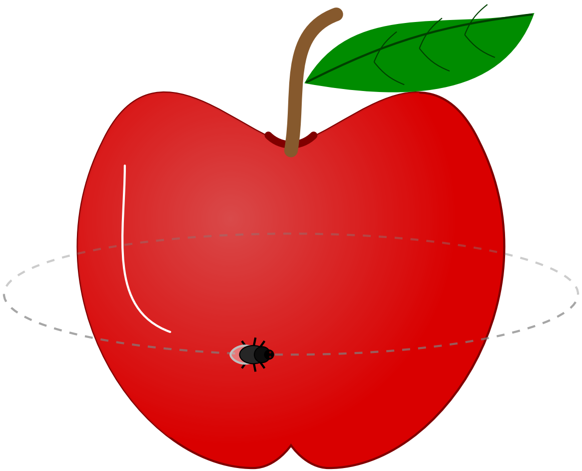
Kalikow's $[T, T^{-1}]$ system





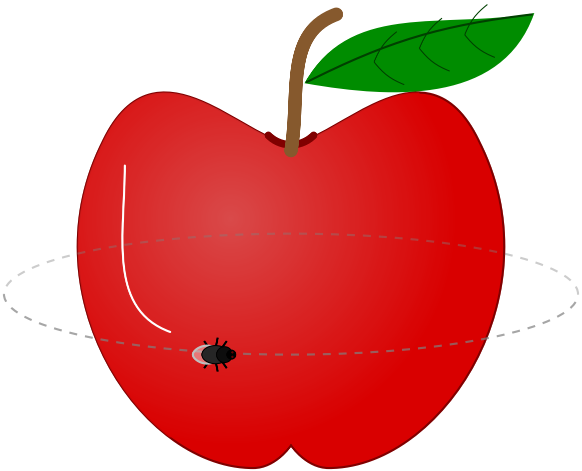
Introduction

Kalikow's $[T, T^{-1}]$ system



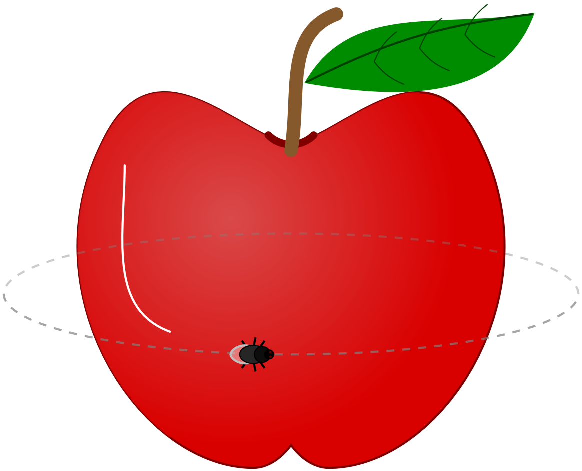
Introduction

Kalikow's $[T, T^{-1}]$ system



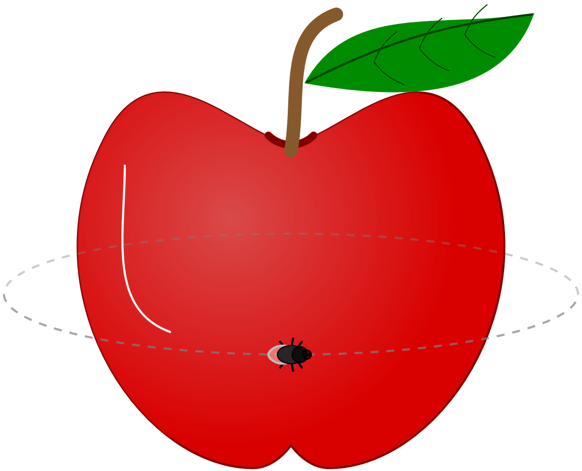
Introduction

Kalikow's $[T, T^{-1}]$ system



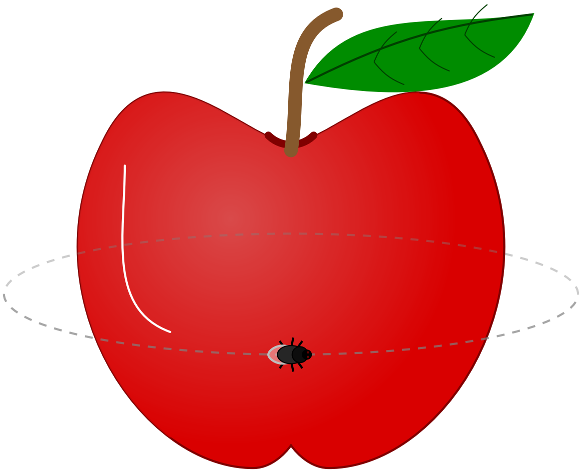
Introduction

Kalikow's $[T, T^{-1}]$ system



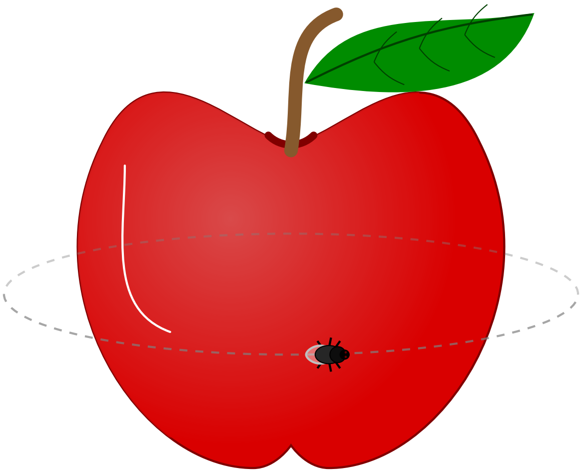
Introduction

Kalikow's $[T, T^{-1}]$ system



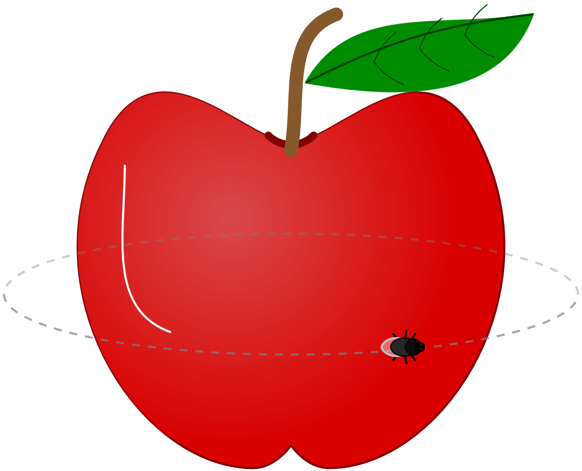
Introduction

Kalikow's $[T, T^{-1}]$ system



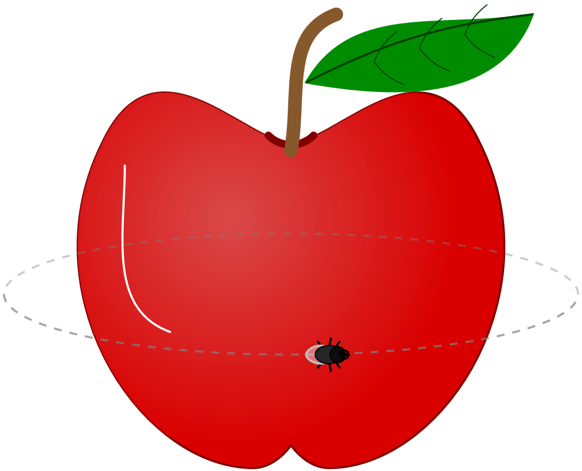
Introduction

Kalikow's $[T, T^{-1}]$ system



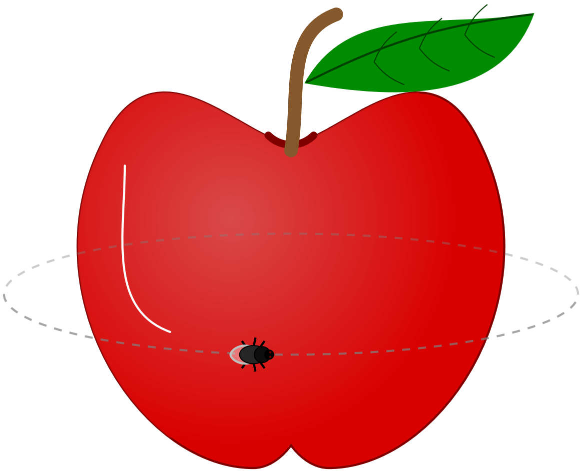
Introduction

Kalikow's $[T, T^{-1}]$ system



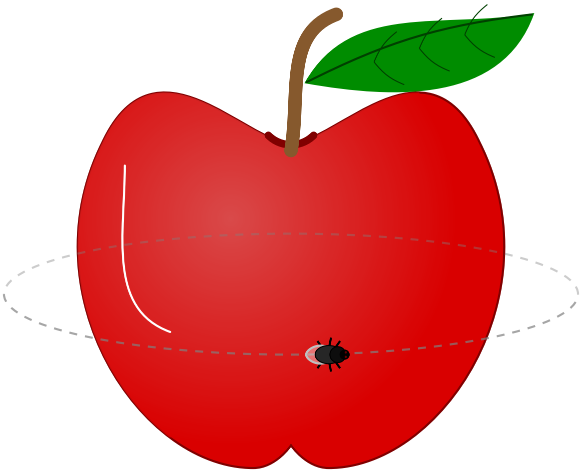
Introduction

Kalikow's $[T, T^{-1}]$ system



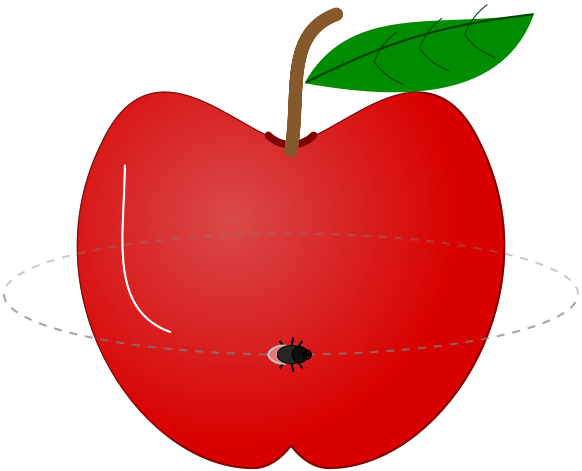
Introduction

Kalikow's $[T, T^{-1}]$ system



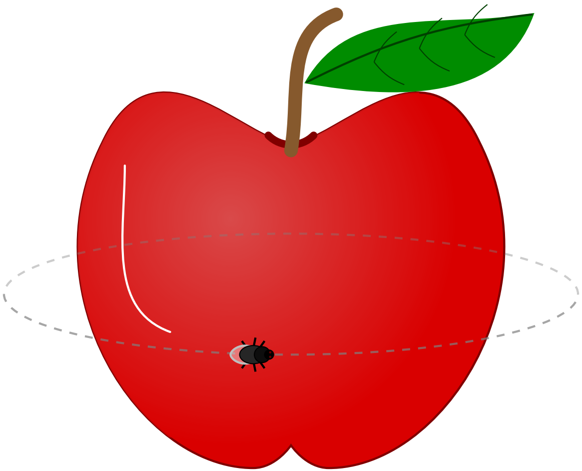
Introduction

Kalikow's $[T, T^{-1}]$ system



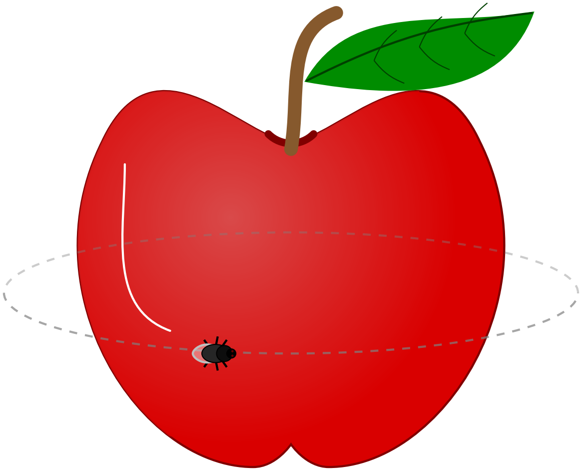
Introduction

Kalikow's $[T, T^{-1}]$ system



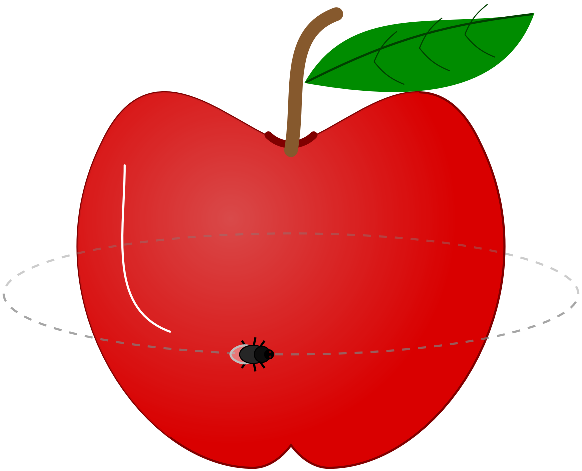
INTRODUCTION

KALIKOW'S $[T, T^{-1}]$ SYSTEM



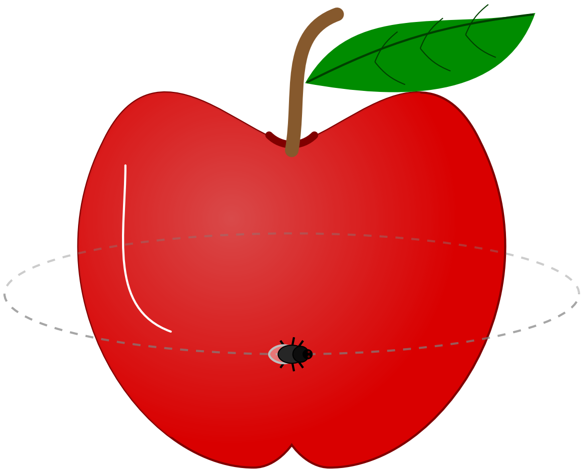
Introduction

Kalikow's $[T, T^{-1}]$ system



Introduction

Kalikow's $[T, T^{-1}]$ system



Fun Fact

Remark

Some insects really move more or less randomly, or according a distribution called Levy walk, or reproduce more or less randomly.

- "Random Search Wired Into Animals May Help Them Hunt" in Quanta magazine.
- Popp, S. & Dornhaus, A. Ants combine systematic meandering and correlated random walks when searching for unknown resources. iScience 26, 105916 (2023).
- Curcio, L. et al. Double stochastic resonance in stink bug sexual communication. Sci Rep 16, 256 (2025)

Ingredients

- Fiber: an automorphism $T \curvearrowright (X, \mathcal{B}, \mu)$
- Base: the shift automorphism σ on $(\{-1, 1\}^{\mathbb{Z}}, \mathcal{C}, \nu_{1/2,1/2})$.
- Selector (or coycle): $\tau\colon \{-1, 1\}^{\mathbb{Z}} \to \mathbb{Z}$, $y \mapsto \tau(y) = y(0)$

INTRODUCTION

KALIKOW'S $[T, T^{-1}]$ SYSTEM

Ingredients

- Fiber: an automorphism $T \curvearrowright (X, \mathcal{B}, \mu)$
- Base: the shift automorphism σ on $(\{-1,1\}^{\mathbb{Z}}, \mathcal{C}, \nu_{1/2,1/2})$.
- Selector (or coycle): $\tau\colon \{-1,1\}^{\mathbb{Z}} \to \mathbb{Z}$, $y \mapsto \tau(y) = y(0)$

Construction

The $[T, T^{-1}]$ transformation is the skew product

$$\sigma \rtimes_\tau T \curvearrowright (\{-1,1\}^{\mathbb{Z}} \times X, \mathcal{C} \otimes \mathcal{B}, \nu_{1/2,1/2} \otimes \mu)$$

$$(y, x) \mapsto (\sigma(y), T^{\tau(y)}(x))$$

Theorem (Kalikow, 1982)

If T has positive entropy, then the $[T, T^{-1}]$ system is Kolmogorov, but not Bernoulli or loosely Bernoulli.

Theorem (Kalikow, 1982)

If T has positive entropy, then the $[T, T^{-1}]$ system is Kolmogorov, but not Bernoulli or loosely Bernoulli.

- It is the first natural or easily defined example with these properties.

Theorem (Kalikow, 1982)

If T has positive entropy, then the $[T, T^{-1}]$ system is Kolmogorov, but not Bernoulli or loosely Bernoulli.

- It is the first natural or easily defined example with these properties.
- For some time it was unclear if Kalikow proved the existence of many non-isomorphic examples, or not (taking different T's).

Theorem (Leuridan 2023, and previous authors)

If T is an irrational rotation then the $[T, T^{-1}]$ system is Bernoulli.

Theorem (Austin, 2014)

There exists an isomorphism invariant I such that $I([T, T^{-1}]) = h_\mu(T)$ for any choice of T.

Theorem (Austin, 2014)

There exists an isomorphism invariant I such that $I([T, T^{-1}]) = h_\mu(T)$ for any choice of T.

- If $[S, S^{-1}]$ is isomorphic to $[T, T^{-1}]$, then S and T must have the same entropy (not necessarily isomorphic by the previous example).

Theorem (Austin, 2014)

There exists an isomorphism invariant I such that $I([T, T^{-1}]) = h_\mu(T)$ for any choice of T.

- If $[S, S^{-1}]$ is isomorphic to $[T, T^{-1}]$, then S and T must have the same entropy (not necessarily isomorphic by the previous example).
- This was proved first in the case of the non-invertible shift $\sigma \curvearrowright \{-1, 1\}^{\mathbb{N}}$ (Hoffman, Heicklen, and Rudolf, and then generalized by Ball).

Problem (A. Kanigowski)

Are there similar results when the Bernoulli shift is replaced by a rotation?

Ingredients

- Fiber: an automorphism $T \curvearrowright (X, \mathcal{B}, \mu)$
- Base: an irrational rotation R_α on $\mathbb{T} = [0, 1)$ with the Borel sigma-algebra and the Lebesgue or Haar measure m.
- Selector (or coycle): $\tau \colon \mathbb{T} \to \mathbb{Z}$, $\theta \mapsto \mathbf{1}_{[0,1/2)}(\theta) - \mathbf{1}_{[1/2,1)}(\theta)$

Ingredients

- Fiber: an automorphism $T \curvearrowright (X, \mathcal{B}, \mu)$
- Base: an irrational rotation R_α on $\mathbb{T} = [0, 1)$ with the Borel sigma-algebra and the Lebesgue or Haar measure m.
- Selector (or coycle): $\tau\colon \mathbb{T} \to \mathbb{Z}$, $\theta \mapsto \mathbf{1}_{[0,1/2)}(\theta) - \mathbf{1}_{[1/2,1)}(\theta)$

Construction

Consider the skew product

$$R_\alpha \rtimes_\tau T \curvearrowright (\mathbb{T} \times X, \mathcal{C} \otimes \mathcal{B}, m \times \mu)$$

$$(\theta, x) \mapsto (R_\alpha(\theta), T^{\tau(\theta)}(x))$$

INTRODUCTION

REMARKS

- The concrete problem is to distinguish $R_\alpha \rtimes_\tau T_1$ and $R_\alpha \rtimes_\tau T_2$ for different T_1 and T_2.

Introduction

Remarks

- The concrete problem is to distinguish $R_\alpha \rtimes_\tau T_1$ and $R_\alpha \rtimes_\tau T_2$ for different T_1 and T_2.
- Entropy gives no information because we always have $h_{m\times\mu}(R_\alpha \rtimes_\tau T) = 0$.

- The concrete problem is to distinguish $R_\alpha \rtimes_\tau T_1$ and $R_\alpha \rtimes_\tau T_2$ for different T_1 and T_2.
- Entropy gives no information because we always have $h_{m\times\mu}(R_\alpha \rtimes_\tau T) = 0$.
- The powers of $R_\alpha \rtimes_\tau T$ are determined to the ergodic sums of $\tau = \mathbf{1}_{[0,1/2)} - \mathbf{1}_{[1/2,1)}$

$$\sum_{i=1}^{n-1} \tau \circ R_\alpha^i$$

Introduction
Remarks

- The concrete problem is to distinguish $R_\alpha \rtimes_\tau T_1$ and $R_\alpha \rtimes_\tau T_2$ for different T_1 and T_2.
- Entropy gives no information because we always have $h_{m\times\mu}(R_\alpha \rtimes_\tau T) = 0$.
- The powers of $R_\alpha \rtimes_\tau T$ are determined to the ergodic sums of $\tau = \mathbf{1}_{[0,1/2)} - \mathbf{1}_{[1/2,1)}$

$$\sum_{i=1}^{n-1} \tau \circ R_\alpha^i$$

in the sense that

$$(R_\alpha \rtimes_\tau T)^n(\theta, x) = (R_\alpha^n(\theta), T^{\sum_{i=1}^{n-1} \tau(R_\alpha^i(\theta))}(x)$$

INTRODUCTION
REMARKS

- The concrete problem is to distinguish $R_\alpha \rtimes_\tau T_1$ and $R_\alpha \rtimes_\tau T_2$ for different T_1 and T_2.
- Entropy gives no information because we always have $h_{m\times\mu}(R_\alpha \rtimes_\tau T) = 0$.
- The powers of $R_\alpha \rtimes_\tau T$ are determined to the ergodic sums of $\tau = \mathbf{1}_{[0,1/2)} - \mathbf{1}_{[1/2,1)}$

$$\sum_{i=1}^{n-1} \tau \circ R_\alpha^i$$

  in the sense that

$$(R_\alpha \rtimes_\tau T)^n(\theta, x) = (R_\alpha^n(\theta), T^{\sum_{i=1}^{n-1} \tau(R_\alpha^i(\theta))}(x)$$

- The sequence $\sum_{i=1}^{n-1} \tau \circ R_\alpha^i$ of measurable functions $\mathbb{T} \to \mathbb{Z}$ is sometimes called deterministic random walk and has nice properties.

Introduction
Remarks

There is abundant literature comparing the statistics of the “deterministic walk”

$$\sum_{i=0}^{n-1} \tau \circ R_\alpha^i \quad : (\mathbb{T}, m) \to \mathbb{R}$$

to the standard random walk

$$\sum_{i=0}^{n-1} \tau \circ \sigma^i \quad : (\{-1, 1\}^{\mathbb{Z}}, \nu_{1/2,1/2}) \to \mathbb{R}$$

Introduction

Remarks

There is abundant literature comparing the statistics of the "deterministic walk"

$$\sum_{i=0}^{n-1} \tau \circ R_\alpha^i \quad : (\mathbb{T}, m) \to \mathbb{R}$$

to the standard random walk

$$\sum_{i=0}^{n-1} \tau \circ \sigma^i \quad : (\{-1, 1\}^{\mathbb{Z}}, \nu_{1/2,1/2}) \to \mathbb{R}$$

(Too much to say)

Let $T\colon [0,1] \to [0,1],\ x \mapsto \sqrt{x}$

Introduction

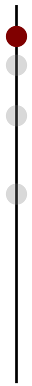
Visualization

Let $T\colon [0,1] \to [0,1]$, $x \mapsto \sqrt{x}$

Introduction
Visualization

Let $T\colon [0,1] \to [0,1],\ x \mapsto \sqrt{x}$

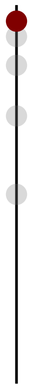
Let $T\colon [0,1] \to [0,1]$, $x \mapsto \sqrt{x}$



Introduction

Visualization

Let $T\colon [0,1] \to [0,1]$, $x \mapsto \sqrt{x}$



Introduction

Visualization

Let $T\colon [0,1] \to [0,1]$, $x \mapsto \sqrt{x}$

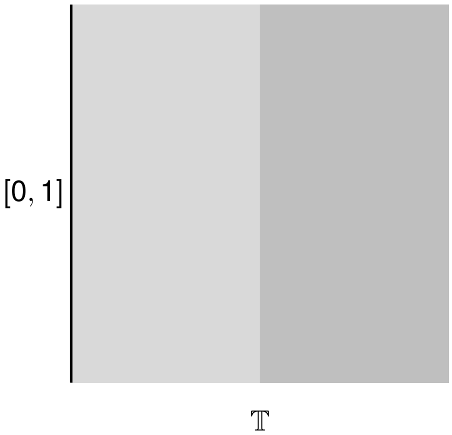
Introduction
Visualization

Let $T\colon [0,1] \to [0,1],\, x \mapsto \sqrt{x}$

Introduction

Visualization

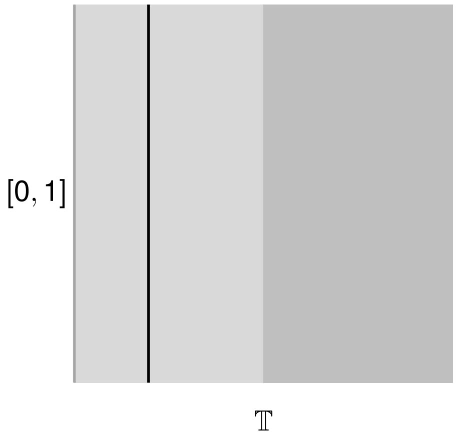
Visualization of $R_\alpha \rtimes_\tau T \curvearrowright [0,1]^2$ with $T(x) = \sqrt{x}$ on $[0,1]$ ($\alpha = 1/\sqrt{26}$)



INTRODUCTION

VISUALIZATION

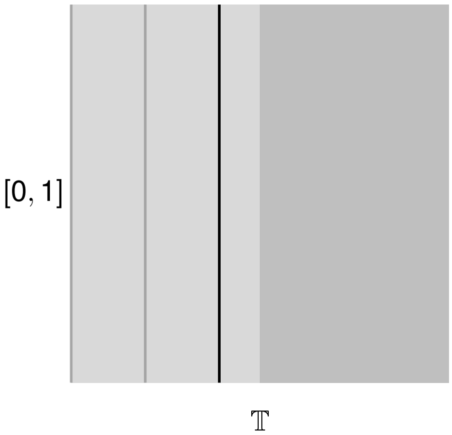
Visualization of $R_\alpha \rtimes_\tau T \curvearrowright [0,1]^2$ with $T(x) = \sqrt{x}$ on $[0,1]$ ($\alpha = 1/\sqrt{26}$)



Introduction

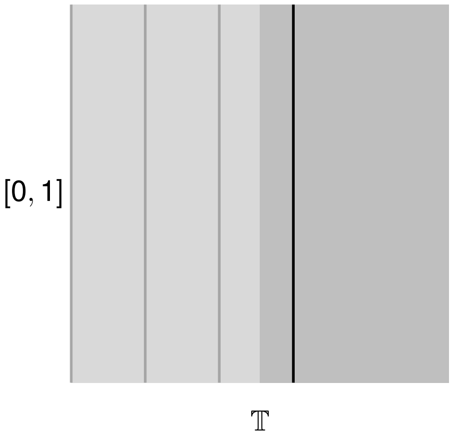
Visualization

Visualization of $R_\alpha \rtimes_\tau T \curvearrowright [0,1]^2$ with $T(x) = \sqrt{x}$ on $[0,1]$ ($\alpha = 1/\sqrt{26}$)



Introduction
Visualization

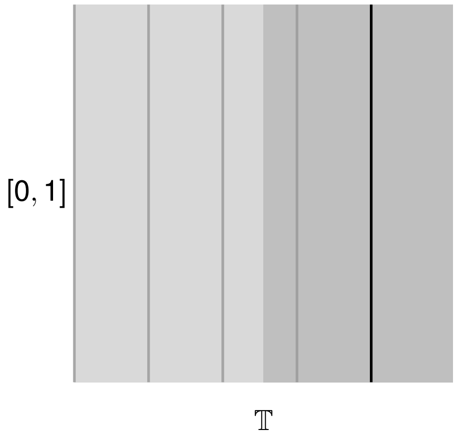
Visualization of $R_\alpha \rtimes_\tau T \curvearrowright [0,1]^2$ with $T(x) = \sqrt{x}$ on $[0,1]$ ($\alpha = 1/\sqrt{26}$)



Introduction

Visualization

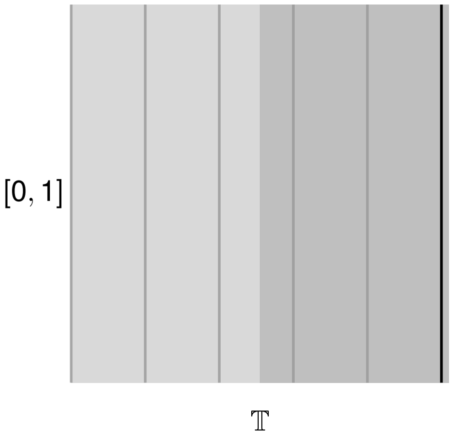
Visualization of $R_\alpha \rtimes_\tau T \curvearrowright [0,1]^2$ with $T(x) = \sqrt{x}$ on $[0,1]$ ($\alpha = 1/\sqrt{26}$)



Introduction

Visualization

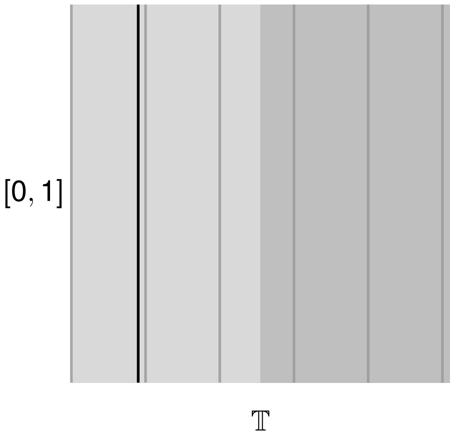
Visualization of $R_\alpha \rtimes_\tau T \curvearrowright [0,1]^2$ with $T(x) = \sqrt{x}$ on $[0,1]$ ($\alpha = 1/\sqrt{26}$)



Introduction

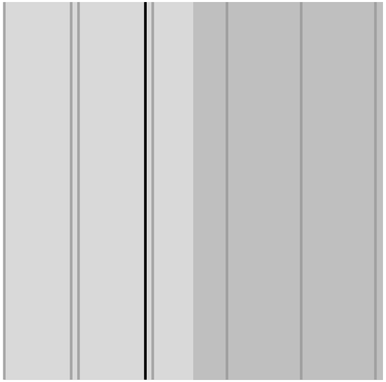
Visualization

Visualization of $R_\alpha \rtimes_\tau T \curvearrowright [0,1]^2$ with $T(x) = \sqrt{x}$ on $[0,1]$ ($\alpha = 1/\sqrt{26}$)



Visualization of $R_\alpha \rtimes_\tau T \curvearrowright [0,1]^2$ with $T(x) = \sqrt{x}$ on $[0,1]$ ($\alpha = 1/\sqrt{26}$)

$[0,1]$

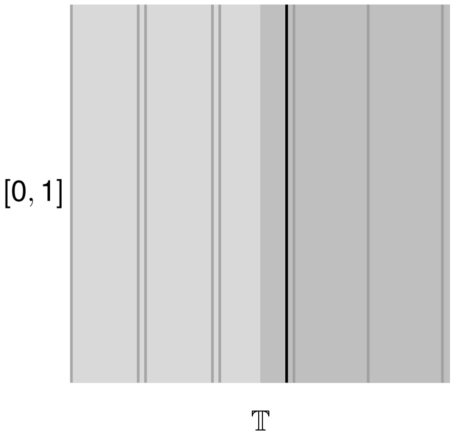


$\mathbb{T}$

INTRODUCTION

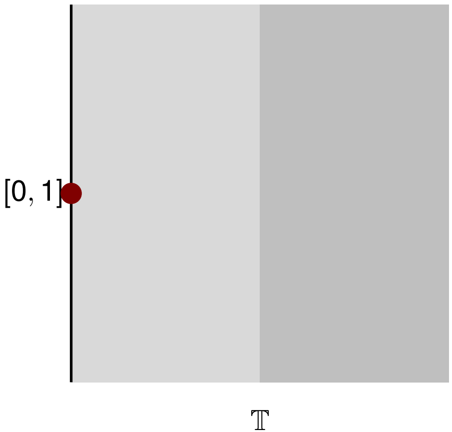
VISUALIZATION

Visualization of $R_\alpha \rtimes_\tau T \curvearrowright [0,1]^2$ with $T(x) = \sqrt{x}$ on $[0,1]$ ($\alpha = 1/\sqrt{26}$)



Introduction
Visualization

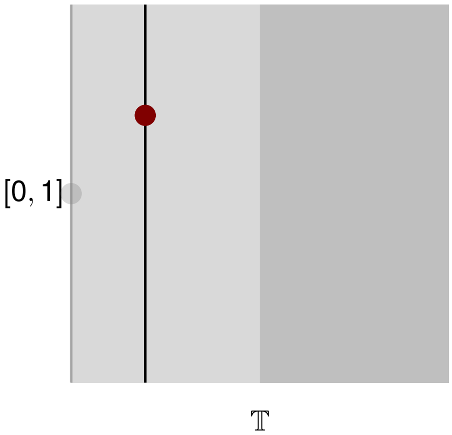
Visualization of $R_\alpha \rtimes_\tau T \curvearrowright [0,1]^2$ with $T(x) = \sqrt{x}$ on $[0,1]$ ($\alpha = 1/\sqrt{26}$)



Introduction

Visualization

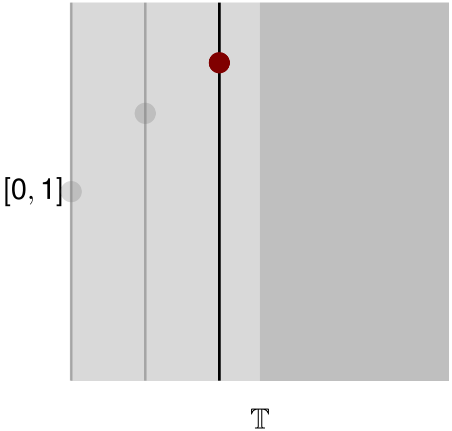
Visualization of $R_\alpha \rtimes_\tau T \curvearrowright [0,1]^2$ with $T(x) = \sqrt{x}$ on $[0,1]$ ($\alpha = 1/\sqrt{26}$)



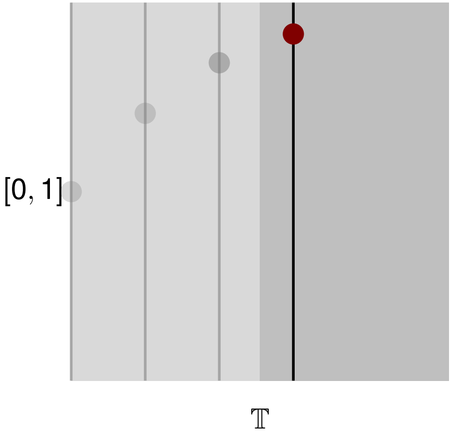
Introduction

Visualization

Visualization of $R_\alpha \rtimes_\tau T \curvearrowright [0,1]^2$ with $T(x) = \sqrt{x}$ on $[0,1]$ ($\alpha = 1/\sqrt{26}$)



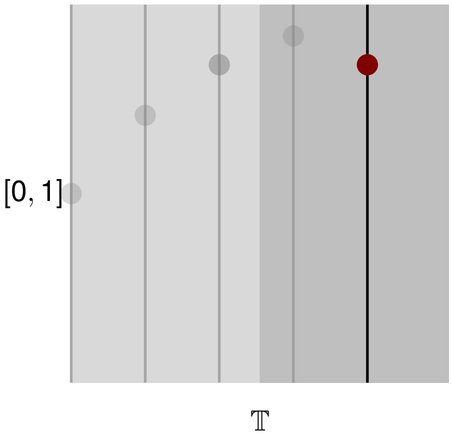
Visualization of $R_\alpha \rtimes_\tau T \curvearrowright [0,1]^2$ with $T(x) = \sqrt{x}$ on $[0,1]$ ($\alpha = 1/\sqrt{26}$)



INTRODUCTION

VISUALIZATION

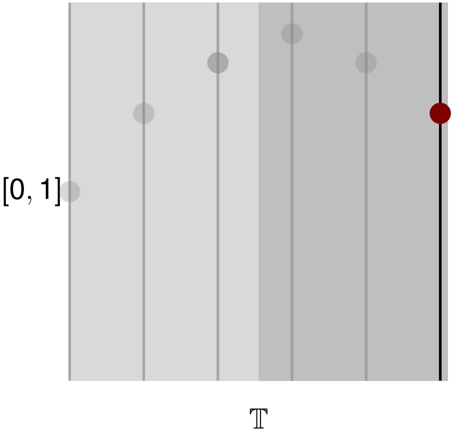
Visualization of $R_\alpha \rtimes_\tau T \curvearrowright [0,1]^2$ with $T(x) = \sqrt{x}$ on $[0,1]$ ($\alpha = 1/\sqrt{26}$)



Introduction

Visualization

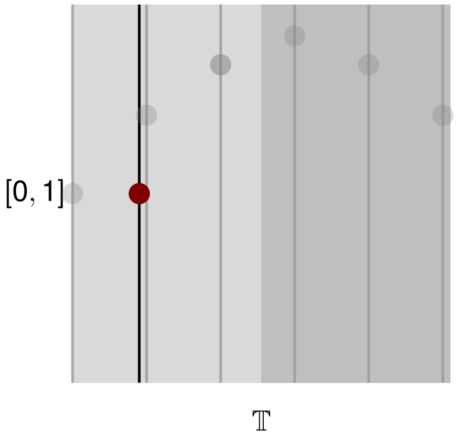
Visualization of $R_\alpha \rtimes_\tau T \curvearrowright [0,1]^2$ with $T(x) = \sqrt{x}$ on $[0,1]$ ($\alpha = 1/\sqrt{26}$)



Introduction

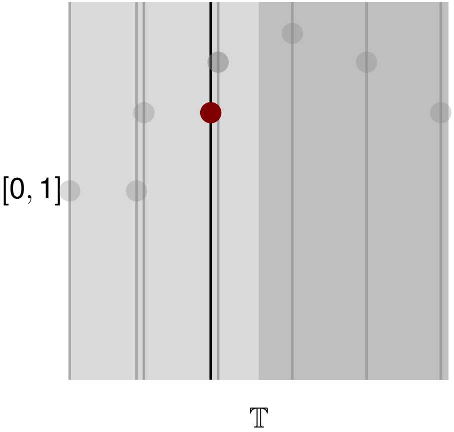
Visualization

Visualization of $R_\alpha \rtimes_\tau T \curvearrowright [0,1]^2$ with $T(x) = \sqrt{x}$ on $[0,1]$ ($\alpha = 1/\sqrt{26}$)



Introduction
Visualization

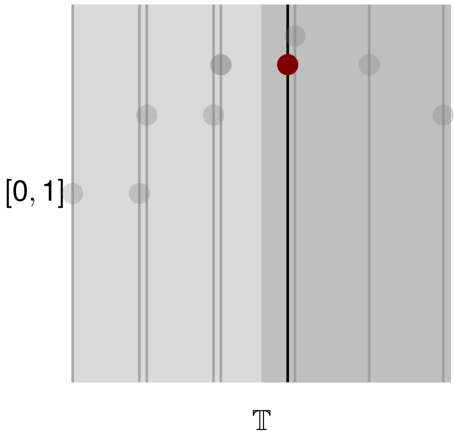
Visualization of $R_\alpha \rtimes_\tau T \curvearrowright [0,1]^2$ with $T(x) = \sqrt{x}$ on $[0,1]$ ($\alpha = 1/\sqrt{26}$)



Introduction

Visualization

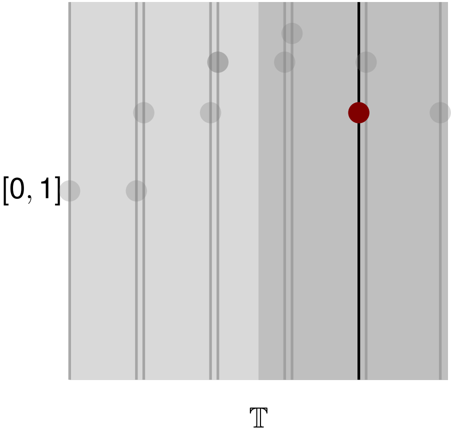
Visualization of $R_\alpha \rtimes_\tau T \curvearrowright [0,1]^2$ with $T(x) = \sqrt{x}$ on $[0,1]$ ($\alpha = 1/\sqrt{26}$)



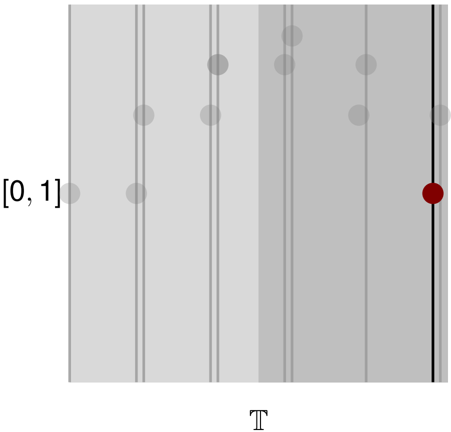
Introduction

Visualization

Visualization of $R_\alpha \rtimes_\tau T \curvearrowright [0,1]^2$ with $T(x) = \sqrt{x}$ on $[0,1]$ ($\alpha = 1/\sqrt{26}$)



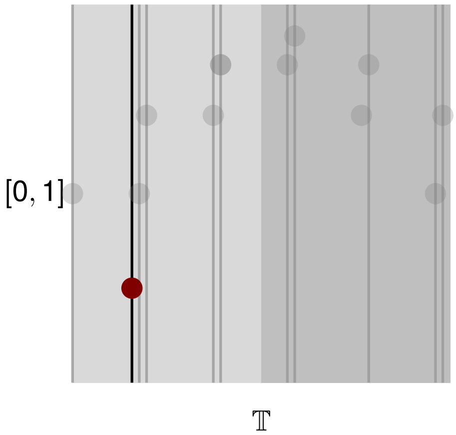
Visualization of $R_\alpha \rtimes_\tau T \curvearrowright [0,1]^2$ with $T(x) = \sqrt{x}$ on $[0,1]$ ($\alpha = 1/\sqrt{26}$)



Introduction

Visualization

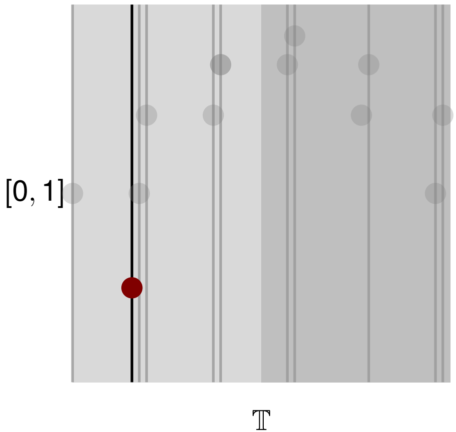
Visualization of $R_\alpha \rtimes_\tau T \curvearrowright [0,1]^2$ with $T(x) = \sqrt{x}$ on $[0,1]$ ($\alpha = 1/\sqrt{26}$)



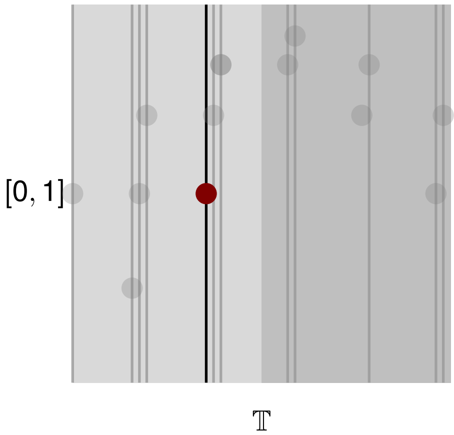
Introduction

Visualization

Visualization of $R_\alpha \rtimes_\tau T \curvearrowright [0,1]^2$ with $T(x) = \sqrt{x}$ on $[0,1]$ ($\alpha = 1/\sqrt{26}$)



Visualization of $R_\alpha \rtimes_\tau T \curvearrowright [0,1]^2$ with $T(x) = \sqrt{x}$ on $[0,1]$ ($\alpha = 1/\sqrt{26}$)



Results

Under mild conditions we have

RESULTS

Under mild conditions we have

- ✓ results analogous to the existing ones for $[T, T^{-1}]$ systems (proved by Austin, Hoffman, Heicklen, and Rudolf, and then generalized by Ball);

Results

Under mild conditions we have

- ✓ results analogous to the existing ones for $[T, T^{-1}]$ systems (proved by Austin, Hoffman, Heicklen, and Rudolf, and then generalized by Ball);
- ✓ also stronger results.

Flip isomorphisms

Two systems T and S are flip isomorphic when T is isomorphic to S or S^{-1}.

Proposition

If T and S are isomorphic then $R_\alpha \rtimes_\tau S$ is isomorphic to $R_\alpha \rtimes_\tau T$.

Proposition

For any T we have that $R_\alpha \rtimes_\tau T$ is isomorphic to $R_\alpha \rtimes_\tau (T^{-1})$

Proposition

For any T we have that $R_\alpha \rtimes_\tau T$ is isomorphic to $R_\alpha \rtimes_\tau (T^{-1})$

Proof.

The map $(\theta, x) \mapsto (\theta + 1/2, x)$ defines an isomorphism, as $\tau(\theta + 1/2) = -\tau(\theta)$. □

Proposition

For any T we have that $R_\alpha \rtimes_\tau T$ is isomorphic to $R_\alpha \rtimes_\tau (T^{-1})$

Proof.

The map $(\theta, x) \mapsto (\theta + 1/2, x)$ defines an isomorphism, as $\tau(\theta + 1/2) = -\tau(\theta)$. □

Corollary

If T and S are *flip* isomorphic then $R_\alpha \rtimes_\tau S$ is isomorphic to $R_\alpha \rtimes_\tau T$.

RESULTS

THE SKEW PRODUCT REMEMBERS THE FLIP ISOMORPHISM CLASS OF THE FIBER

Theorem 1

Let S and T be ergodic automorphisms of $(X, \mathcal{B}, \mu)$ with $S^2 = S \circ S$ also ergodic.

Results

The skew product remembers the flip isomorphism class of the fiber

Theorem 1

Let S and T be ergodic automorphisms of $(X, \mathcal{B}, \mu)$ with $S^2 = S \circ S$ also ergodic. Then the following are equivalent:

- $R_\alpha \rtimes_\tau S$ *is isomorphic to* $R_\alpha \rtimes_\tau T$
- S *is flip isomorphic to* T

Results

The skew product remembers the flip isomorphism class of the fiber

Theorem 1

Let S and T be ergodic automorphisms of $(X, \mathcal{B}, \mu)$ with $S^2 = S \circ S$ also ergodic. Then the following are equivalent:

- $R_\alpha \rtimes_\tau S$ *is isomorphic to* $R_\alpha \rtimes_\tau T$
- S *is flip isomorphic to* T

This is surprising when contrasted to the following.

Example (due to M. Lemańczyk, see arxiv:2509.09003)

For weakly mixing S and T, it is possible that $R_\alpha \times S$ is isomorphic $R_\alpha \times T$ but S not flip isomorphic to T.

Results

The skew product remembers the flip isomorphism class of the fiber

Theorem 2

Let S and T be ergodic automorphisms of $(X, \mathcal{B}, \mu)$ with $S^2 = S \circ S$ also ergodic. Suppose further that S is not a rotation (on a compact abelian group with its Haar measure, up to isomorphism).

RESULTS

THE SKEW PRODUCT REMEMBERS THE FLIP ISOMORPHISM CLASS OF THE FIBER

Theorem 2

Let S and T be ergodic automorphisms of $(X, \mathcal{B}, \mu)$ with $S^2 = S \circ S$ also ergodic. Suppose further that S is not a rotation (on a compact abelian group with its Haar measure, up to isomorphism).Then every isomorphism Υ between $R_\alpha \rtimes_\tau S$ and $R_\alpha \rtimes_\tau T$, if exists, has one of the following two forms:

Theorem 2

Let S and T be ergodic automorphisms of $(X, \mathcal{B}, \mu)$ with $S^2 = S \circ S$ also ergodic. Suppose further that S is not a rotation (on a compact abelian group with its Haar measure, up to isomorphism).Then every isomorphism Υ between $R_\alpha \rtimes_\tau S$ and $R_\alpha \rtimes_\tau T$, if exists, has one of the following two forms:

1. $\Upsilon(\theta, x) = (\theta, \upsilon(x))$ *where υ is an isomorphism between S and T.*
2. $\Upsilon(\theta, x) = (\theta + 1/2, \upsilon(x))$ *where υ is an isomorphism between S and T^{-1}.*

Theorem 2

Let S and T be ergodic automorphisms of $(X, \mathcal{B}, \mu)$ with $S^2 = S \circ S$ also ergodic. Suppose further that S is not a rotation (on a compact abelian group with its Haar measure, up to isomorphism).Then every isomorphism Υ between $R_\alpha \rtimes_\tau S$ and $R_\alpha \rtimes_\tau T$, if exists, has one of the following two forms:

1. $\Upsilon(\theta, x) = (\theta, \upsilon(x))$ *where υ is an isomorphism between S and T.*
2. $\Upsilon(\theta, x) = (\theta + 1/2, \upsilon(x))$ *where υ is an isomorphism between S and T^{-1}.*

- Choosing $S = T$ this result computes the so called centralizer of $R_\alpha \rtimes_\tau T$.

RESULTS

AN APPLICATION TO CLASSIFICATION

Since the function

$$T \mapsto R_\alpha \rtimes_\tau T$$

is Borel

Results

An application to classification

Since the function

$$T \mapsto R_\alpha \rtimes_\tau T$$

is Borel, we have a Borel reduction

$$\Big(\text{Flip isomorphism problem for } \; T, \;\; \text{with } T^2 \text{ ergodic}\Big) \leq_{\mathcal{B}} (\text{Isomorphism problem for } \; R_\alpha \rtimes_\tau T)$$

Results

An application to classification

Since the function

$$T \mapsto R_\alpha \rtimes_\tau T$$

is Borel, we have a Borel reduction

$$\Big(\text{Flip isomorphism problem for } \; T, \;\; \text{with } T^2 \text{ ergodic}\Big) \leq_{\mathcal{B}} (\text{Isomorphism problem for } \; R_\alpha \rtimes_\tau T)$$

- The problem at the left is maximally difficult (Kunde, 2024)

Results

An application to classification

Since the function

$$T \mapsto R_\alpha \rtimes_\tau T$$

is Borel, we have a Borel reduction

$$\Big(\text{Flip isomorphism problem for } \ T, \ \text{ with } T^2 \text{ ergodic}\Big) \leq_{\mathcal{B}} (\text{Isomorphism problem for } \ R_\alpha \rtimes_\tau T)$$

- The problem at the left is maximally difficult (Kunde, 2024)
- Thus the problem at the right is maximally difficult.

RESULTS

AN APPLICATION TO CLASSIFICATION

Since the function

$$T \mapsto R_\alpha \rtimes_\tau T$$

is Borel, we have a Borel reduction

$$\Big(\text{Flip isomorphism problem for } T, \ \text{ with } T^2 \text{ ergodic}\Big) \leq_{\mathcal{B}} (\text{Isomorphism problem for } R_\alpha \rtimes_\tau T)$$

- The problem at the left is maximally difficult (Kunde, 2024)
- Thus the problem at the right is maximally difficult.
- This can be used to show that the isomorphism problem is difficult with some slowness restrictions (choosing α):
  - Having zero slow entropy at any prescribed scale ✓.

Method

1. First we imagine our measurable functions are continuous.

Proof ideas

Key ideas

Method

1. First we imagine our measurable functions are continuous.
2. We do trickery

Method

1. First we imagine our measurable functions are continuous.
2. We do trickery
3. Then we invoke Lusin Theorem

Method

1. First we imagine our measurable functions are continuous.
2. We do trickery
3. Then we invoke Lusin Theorem(which says that measurable functions are continuous on a set with big measure)

Proof ideas

Key ideas

Consider the group of automorphisms of $(X, \mathcal{B}, \mu)$

$$\mathrm{Aut}(X, \mathcal{B}, \mu)$$

PROOF IDEAS

KEY IDEAS

Consider the group of automorphisms of $(X, \mathcal{B}, \mu)$

$$\mathrm{Aut}(X, \mathcal{B}, \mu)$$

This is a Polish group with the weak topology :

PROOF IDEAS

KEY IDEAS

Consider the group of automorphisms of $(X, \mathcal{B}, \mu)$

$$\mathrm{Aut}(X, \mathcal{B}, \mu)$$

This is a Polish group with the weak topology : $(f_n)_{n \in \mathbb{N}}$ converges to f if for all $B \in \mathcal{B}$

$$\mu(f_n^{-1}(B) \triangle f^{-1}(B)) \to 0$$

PROOF IDEAS

KEY IDEAS

Consider the group of automorphisms of $(X, \mathcal{B}, \mu)$

$$\mathrm{Aut}(X, \mathcal{B}, \mu)$$

This is a Polish group with the weak topology : $(f_n)_{n\in\mathbb{N}}$ converges to f if for all $B \in \mathcal{B}$

$$\mu(f_n^{-1}(B) \triangle f^{-1}(B)) \to 0$$

Proof ideas

Proof outline

Fix S and T with S^2 ergodic, and assume that there exists an isomorphism from $R_\alpha \rtimes_\tau S$ to $R_\alpha \rtimes_\tau T$

$$\Upsilon : \mathbb{T} \times X \to \mathbb{T} \times X$$

$$\Upsilon(\theta, x) = (\phi(\theta, x), \psi(\theta, x))$$

Fix S and T with S^2 ergodic, and assume that there exists an isomorphism from $R_\alpha \rtimes_\tau S$ to $R_\alpha \rtimes_\tau T$

$$\Upsilon : \mathbb{T} \times X \to \mathbb{T} \times X$$

$$\Upsilon(\theta, x) = (\phi(\theta, x), \psi(\theta, x))$$

One proves:

1. $\phi(\theta, x)$ depends only on θ (up to a null set)

Proof ideas
Proof outline

Fix S and T with S^2 ergodic, and assume that there exists an isomorphism from $R_\alpha \rtimes_\tau S$ to $R_\alpha \rtimes_\tau T$

$$\Upsilon : \mathbb{T} \times X \to \mathbb{T} \times X$$

$$\Upsilon(\theta, x) = (\phi(\theta, x), \psi(\theta, x))$$

One proves:

1. $\phi(\theta, x)$ depends only on θ (up to a null set)
2. $\phi(\theta, x) = \theta + c$ for some $c \in \mathbb{T}$.

Fix S and T with S^2 ergodic, and assume that there exists an isomorphism from $R_\alpha \rtimes_\tau S$ to $R_\alpha \rtimes_\tau T$

$$\Upsilon\colon \mathbb{T} \times X \to \mathbb{T} \times X$$

$$\Upsilon(\theta, x) = (\phi(\theta, x), \psi(\theta, x))$$

One proves:

1. $\phi(\theta, x)$ depends only on θ (up to a null set)
2. $\phi(\theta, x) = \theta + c$ for some $c \in \mathbb{T}$.
3. The map

$$\psi_\theta\colon X \to X, x \mapsto \psi(\theta, x)$$

   is invertible and measure preserving for almost every θ.

Fix S and T with S^2 ergodic, and assume that there exists an isomorphism from $R_\alpha \rtimes_\tau S$ to $R_\alpha \rtimes_\tau T$

$$\Upsilon \colon \mathbb{T} \times X \to \mathbb{T} \times X$$

$$\Upsilon(\theta, x) = (\phi(\theta, x), \psi(\theta, x))$$

One proves:

1. $\phi(\theta, x)$ depends only on θ (up to a null set)
2. $\phi(\theta, x) = \theta + c$ for some $c \in \mathbb{T}$.
3. The map
   $$\psi_\theta \colon X \to X, x \mapsto \psi(\theta, x)$$
   is invertible and measure preserving for almost every θ.
4. The map $\mathbb{T} \to \mathrm{Aut}(X, \mathcal{B}, \mu)$, $\theta \mapsto \psi_\theta$ is measurable.

Proof ideas

Proof outline

Fix S and T with S^2 ergodic, and assume that there exists an isomorphism from $R_\alpha \rtimes_\tau S$ to $R_\alpha \rtimes_\tau T$

$$\Upsilon \colon \mathbb{T} \times X \to \mathbb{T} \times X$$

$$\Upsilon(\theta, x) = (\phi(\theta, x), \psi(\theta, x))$$

One proves:

1. $\phi(\theta, x)$ depends only on θ (up to a null set)
2. $\phi(\theta, x) = \theta + c$ for some $c \in \mathbb{T}$.
3. The map
$$\psi_\theta \colon X \to X, x \mapsto \psi(\theta, x)$$
is invertible and measure preserving for almost every θ.
4. The map $\mathbb{T} \to \mathrm{Aut}(X, \mathcal{B}, \mu)$, $\theta \mapsto \psi_\theta$ is measurable.
5. Either $\psi_\theta \circ S = T \circ \psi_\theta$ or $\psi_\theta \circ S = T^{-1} \circ \psi_\theta$ almost everywhere.

PROOF IDEAS

PROOF OUTLINE

Fix S and T with S^2 ergodic, and assume that there exists an isomorphism from $R_\alpha \rtimes_\tau S$ to $R_\alpha \rtimes_\tau T$

$$\Upsilon \colon \mathbb{T} \times X \to \mathbb{T} \times X$$

$$\Upsilon(\theta, x) = (\phi(\theta, x), \psi(\theta, x))$$

One proves:

1. $\phi(\theta, x)$ depends only on θ (up to a null set)
2. $\phi(\theta, x) = \theta + c$ for some $c \in \mathbb{T}$.
3. The map

   $$\psi_\theta \colon X \to X, x \mapsto \psi(\theta, x)$$

   is invertible and measure preserving for almost every θ.
4. The map $\mathbb{T} \to \mathrm{Aut}(X, \mathcal{B}, \mu)$, $\theta \mapsto \psi_\theta$ is measurable.
5. Either $\psi_\theta \circ S = T \circ \psi_\theta$ or $\psi_\theta \circ S = T^{-1} \circ \psi_\theta$ almost everywhere.
6. Provided S is not a rotation, $\theta \to \psi_\theta$ is constant almost everywhere, and $c \in \{0, 1/2\}$.

Proof ideas

Ingredients

The trick is to write

$$\Upsilon \circ (R_\alpha \rtimes_\tau S)^n = (R_\alpha \rtimes_\tau T)^n \circ \Upsilon$$

and then choose n.

PROOF IDEAS

INGREDIENTS

The trick is to write

$$\Upsilon \circ (R_\alpha \rtimes_\tau S)^n = (R_\alpha \rtimes_\tau T)^n \circ \Upsilon$$

and then choose n.

Proof ideas
Ingredients

Recall that

$$(R_\alpha \rtimes_\tau T)^n(\theta, x) = (\theta + n\alpha, T^{\sum_{i=0}^{n-1} \tau(R_\alpha^i(\theta))}(x))$$

Proof ideas

Ingredients

Recall that

$$(R_\alpha \rtimes_\tau T)^n(\theta, x) = (\theta + n\alpha, T^{\sum_{i=0}^{n-1} \tau(R_\alpha^i(\theta))}(x))$$

Write

$$\tau(n, \theta) = \sum_{i=0}^{n-1} \tau(R_\alpha^i(\theta))$$

Proof ideas
Ingredients

Recall that

$$(R_\alpha \rtimes_\tau T)^n(\theta, x) = (\theta + n\alpha, T^{\sum_{i=0}^{n-1} \tau(R_\alpha^i(\theta))}(x))$$

Write

$$\tau(n, \theta) = \sum_{i=0}^{n-1} \tau(R_\alpha^i(\theta))$$

Proposition (Aaronson and Keane 1982)

Any irrational admits infinitely many rational approximations

$$|\alpha - p_n/q_n| < 1/(2q_n^2)$$

with p_n and q_n coprime, and with q_n **odd.**

PROOF IDEAS
INGREDIENTS

Recall that

$$(R_\alpha \rtimes_\tau T)^n(\theta, x) = (\theta + n\alpha, T^{\sum_{i=0}^{n-1} \tau(R_\alpha^i(\theta))}(x))$$

Write

$$\tau(n, \theta) = \sum_{i=0}^{n-1} \tau(R_\alpha^i(\theta))$$

Proposition (Aaronson and Keane 1982)

Any irrational admits infinitely many rational approximations

$$|\alpha - p_n/q_n| < 1/(2q_n^2)$$

with p_n and q_n coprime, and with q_n **odd**.
For any such sequence, $(\tau(q_n, \theta))_{n\in\mathbb{N}}$ has infinitely many 1's and -1's for almost every $\theta \in \mathbb{T}$.

Proof ideas

Ingredients

Given $\Upsilon \circ (R_\alpha \rtimes_\tau S)^{q_n} = (R_\alpha \rtimes_\tau T)^{q_n} \circ \Upsilon$, we look at the second coordinate and find

$$\psi_{\theta+q_n\alpha} \circ S^{\tau(q_n,\theta)} = T^{\tau(q_n,\theta+c)} \circ \psi_\theta$$

Suppose that θ admits a subsequence $(q_{n_k})_{k\in\mathbb{N}}$ with $\tau(q_{n_k},\theta) = 1$ and $\tau(q_{n_k},\theta + c) = 1$:

PROOF IDEAS

INGREDIENTS

Given $\Upsilon \circ (R_\alpha \rtimes_\tau S)^{q_n} = (R_\alpha \rtimes_\tau T)^{q_n} \circ \Upsilon$, we look at the second coordinate and find

$$\psi_{\theta+q_n\alpha} \circ S^{\tau(q_n,\theta)} = T^{\tau(q_n,\theta+c)} \circ \psi_\theta$$

Suppose that θ admits a subsequence $(q_{n_k})_{k\in\mathbb{N}}$ with $\tau(q_{n_k},\theta) = 1$ and $\tau(q_{n_k},\theta + c) = 1$:

$$\psi_{\theta+q_{n_k}\alpha} \circ S = T \circ \psi_\theta$$

Asuming ψ_θ depends continuously on θ, taking $k \to \infty$ we find

$$\psi_\theta \circ S = T \circ \psi_\theta$$

PROOF IDEAS

INGREDIENTS

Step 6 has a similar flavor: one writes an equation in $\mathrm{Aut}(X, \mathcal{B}, \mu)$ withe elements depending measurably on $\theta \in \mathbb{T}$, iterate q_n times, use Lusin's Theorem.

The end

Thanks![1]

[1] Beamer theme: elegant slides. Tikz pictures: Kimi AI and Gemini AI