TOPOLOGICAL DYNAMICAL SYSTEMS HAVING A COMPUTABLE COPY

SEMINAR TALK FOR CZECH ACADEMY OF SCIENCES, CZECH REPUBLIC

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TALK CONTENTS

HELLO

- Computability theory for uncountable spaces
- An application to a problem from topological dynamics.

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Remark

Please interrupt me all you want

COMPUTABILITY THEORY ON THE COUNTABLE WORLD

COMPUTABLE FUNCTIONS

Definition (informal)

A function $f: \mathbb{N} \to \mathbb{N}$ is computable if some computer program computes it.

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We can replace $\mathbb{N}$ by other countable sets: polynomials, matrices, finitely presented groups, computer programs.

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  (computable for $n = 2$, unknown for $n = 3$)

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- $f\colon \mathbb{Z}[x_1, x_2, \ldots, x_n] \to \{0, 1\}$,

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  Open problem if it is computable or uncomputable.

DETOUR

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Now we make a detour to look into subshifts, which are dynamical systems with fun computable and uncomputable properties.

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Given a rational $\epsilon > 0$, $g(n, \epsilon)$ approximates $f(n)$ with precision ϵ:

$$|g(n, \epsilon) - f(n)| < \epsilon$$

COMPUTABILITY ON POLISH SPACES

THE DEFINITION

Definition

A computable metric space is a tuple $(X, d, (s_i)_{i\in\mathbb{N}})$ of a Polish metric space (X, d) and a countable dense set $(s_i)_{i\in\mathbb{N}}$ such that

$$D\colon \mathbb{N}^2 \to \mathbb{R}$$

$$D(i, j) = d(s_i, s_j)$$

is computable.

COMPUTABILITY ON POLISH SPACES

SOME EXAMPLES OF COMPUTABLE POLISH SPACE

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4. The Cantor space $\{0,1\}^{\mathbb{N}}$ with

$$d(x,y) = \inf\{2^{-n} : x|_{\{0,\ldots,n-1\}} = y|_{\{0,\ldots,n-1\}}\}$$

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7. Your favorite Polish metric spaces (with probability 99%)

COMPUTABILITY ON POLISH SPACES

WHAT CAN WE DO WITH COMPUTABLE METRIC SPACES?

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- An set U is computably open if we can write

$$U = \bigcup_{i\in\mathbb{N}} B(s_{f(i)}, r(i))$$

where $f\colon \mathbb{N} \to \mathbb{N}$ and $r\colon \mathbb{N} \to \mathbb{Q}$ are computable.

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- A computable closed set is the complement of a computable open set.
- A function $F\colon X \to X$ is computable if
  - The preimage of a computably open set is computably open (computably on f and r).
  - F is continuous and there is a computable $f\colon \mathbb{N}^2 \to \mathbb{N}$ with

    $$d(F(s_n), s_{f(n,m)}) < 2^{-m}$$

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There is a computable function $f\colon [0,1] \to \mathbb{R}$ which is $\mathcal{C}^1$, but whose derivative f' is uncomputable.

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WHAT PEOPLE LIKE TO DO

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TOPOLOGICAL DYNAMICAL SYSTEMS

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- What is a dynamical system?

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Definition (more general)

Let G be a finitely generated group. A topological dynamical system is a pair (X, T) of a compact metric space X and a continuous action $T\colon G \times X \to X$, $(g, x) \mapsto T^g(x)$.

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TOPOLOGICAL DYNAMICAL SYSTEMS

SUBSHIFTS

Subshifts are a class of topological dynamical systems.

TOPOLOGICAL DYNAMICAL SYSTEMS

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Subshifts are a class of topological dynamical systems. Some examples

TOPOLOGICAL DYNAMICAL SYSTEMS

SUBSHIFTS

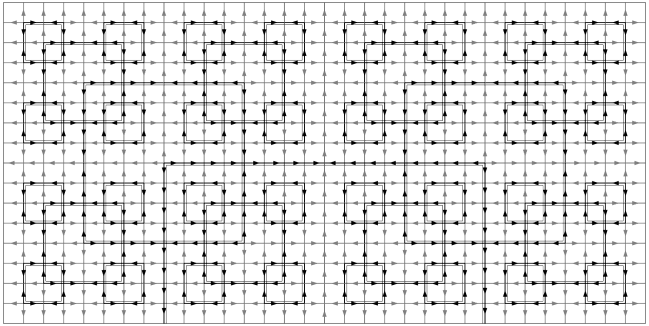

Figure. A subshift

TOPOLOGICAL DYNAMICAL SYSTEMS

SUBSHIFTS

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TOPOLOGICAL DYNAMICAL SYSTEMS

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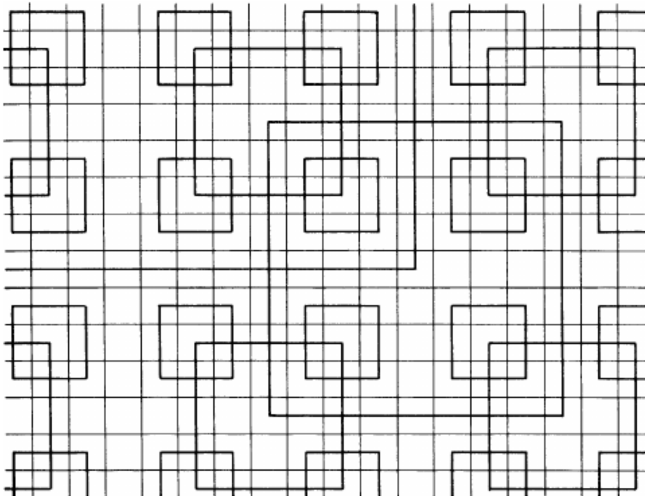

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TOPOLOGICAL DYNAMICAL SYSTEMS

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Let G be a finitely generated group and let $\mathcal{C}$ be a finite set of colors.The fullshift is

$$\mathcal{C}^G = \{x \colon G \to \mathcal{C}\}$$

$$G \curvearrowright \mathcal{C}^G, gx(h) = x(g^{-1}h)$$

Definition (subshift)

A subshift is a subset of $\mathcal{C}^G$ which is shift invariant and topologicall closed.

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We can define a subshift by forbidding a finite collection $\mathcal{F}$ of patterns :

$$X_{\mathcal{F}} = \{x \in \mathcal{C}^G : \text{ no } p \in \mathcal{F} \text{ appears in x}\}$$

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These are called subshifts of finite type (SFTs).

TOPOLOGICAL DYNAMICAL SYSTEMS

THE CLASS OF FACTORS OF SFTS

Question

What systems are topological factors of subshifts of finite type?

TOPOLOGICAL DYNAMICAL SYSTEMS

THE CLASS OF FACTORS OF SFTS

Question

What systems are topological factors of subshifts of finite type?

Definition 4.1

Let (X, T) and (Y, S) be two topological systems associated to G. We say that (X, T) is a factor of (Y, S) if there is a continuous surjection $\pi\colon Y \to X$ intertwining the actions:

$$\pi \circ S^g = T^g \circ \pi \quad g \in G$$

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It turns out that for some groups, one can prove a sufficiency criterion using computable dynamical systems.

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We say that a topological dynamical system (X, T) associated to G is computable if all the objects can be made computable (up to replacing (X, T) by an isomorphic copy):

COMPUTABLE TOPOLOGICAL DYNAMICAL SYSTEMS

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- X computably compact subset of a computable metric space.
- $T^g \colon X \to X$ computable for all $g \in G$.

COMPUTABLE TOPOLOGICAL DYNAMICAL SYSTEMS

EXAMPLES:

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- Irrational rotations with a computable angle.

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- Irrational rotations with a computable angle.
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- Your favorite topological dynamical systems (chance 50%)

COMPUTABLE TOPOLOGICAL DYNAMICAL SYSTEMS

A CLASSIC THEOREM

Theorem (folklore)

Every topological dynamical system is the topological factor of one with zero topological dimension.

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Theorem (with S. Barbieri and C. Rojas, 2025)

For a recursively presented group, every computable dynamical system is the topological factor of a computable and zero dimensional topological dynamical system.

COMPUTABLE TOPOLOGICAL DYNAMICAL SYSTEMS

SELF-SIMULABLE GROUPS

Theorem (S. Barbieri, M. Sablik, V. Salo)

There exists a class of nonamenable recursively presented groups such that every zero-dimensional topological system is the factor of a subshift of finite type (*self-simulable groups*).

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- $G \times H$, with G and H nonamenable.
- $GL_n(\mathbb{Z})$ (invertible $n \times n$ matrices) with $n \geq 5$.

COMPUTABLE TOPOLOGICAL DYNAMICAL SYSTEMS

NEW EXAMPLES OF FACTORS OF SUBSHIFTS OF FINITE TYPE

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1. The action $GL_n(\mathbb{Z}) \curvearrowright \mathbb{R}^n/\mathbb{Z}^n$ by left matrix multiplication ($n \geq 5$).

COMPUTABLE TOPOLOGICAL DYNAMICAL SYSTEMS

NEW EXAMPLES OF FACTORS OF SUBSHIFTS OF FINITE TYPE

1. The action $GL_n(\mathbb{Z}) \curvearrowright \mathbb{R}^n/\mathbb{Z}^n$ by left matrix multiplication ($n \geq 5$).
2. Every computable dynamical system of $F_2 \times F_2$.

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NEW EXAMPLES OF FACTORS OF SUBSHIFTS OF FINITE TYPE

1. The action $GL_n(\mathbb{Z}) \curvearrowright \mathbb{R}^n/\mathbb{Z}^n$ by left matrix multiplication ($n \geq 5$).
2. Every computable dynamical system of $F_2 \times F_2$.
3. Every finitely presented algebraic action of $F_2 \times F_2$.
4. ...

PROOF IDEAS

Proof ideas : let's take $G = \mathbb{Z}$ for simplicity.

PROOF IDEAS

Thanks!

Figure. Cat caption