The strong topological Rokhlin property and Medvedev degrees of shifts of finite type

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Initial remarks

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3. Talk content: an application of computability to a genericity problem in topological dynamics.

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$\sigma \colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$

Generalized shifts (reversible)

$0 \mapsto 10, 1 \mapsto 01$

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Generically, there is only one Cantor system (up to topological conjugacy).

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Definition

A countable group has the *strong topological Rokhlin property* (STRP) if it admits a generic Cantor system.

I will present a computability obstruction for the existence of a generic Cantor system for some groups.

Definition (for this talk)

Topological dynamical system =

$$G \times X \to X,\ (g, x) \mapsto T^g(x)$$

action T of a finitely generated group G on a compact metrizable space X by homeomorphisms.

Oficially we should always speak of the pair (X, T), but we may omit some of them if this is clear from the context.

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Subshift of finite type $\approx$ space of tilings of a group.

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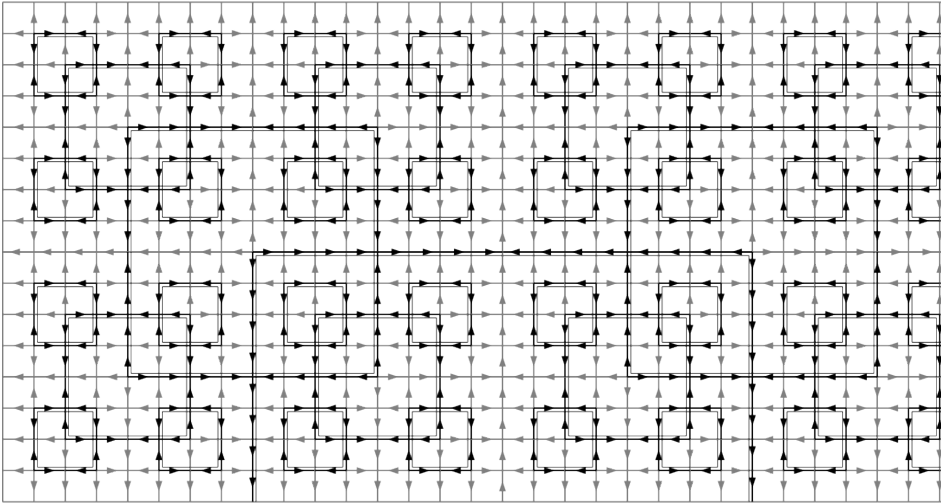

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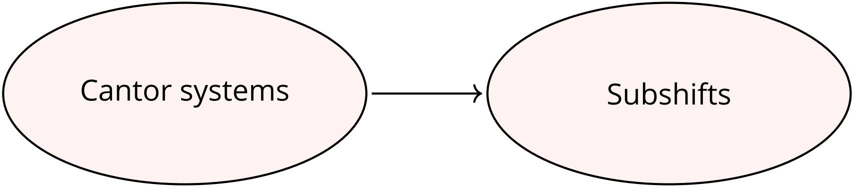
Surprise

Genericity questions for Cantor systems can be reduced to subshifts.



Space of Cantor systems $\to$ space of subshifts

Theorem (Doucha, 2024)

For a finitely generated group G, the following two conditions are equivalent:

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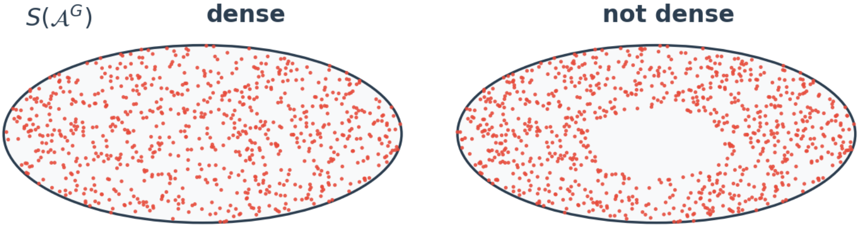
For a finitely generated group G, the following two conditions are equivalent:

1. G admits a generic Cantor system.
2. For every finite alphabet $\mathcal{A}$, in the space of subshifts with alphabet $\mathcal{A}$, projectively isolated subshifts are dense.



Topology for the space of subshifts

For a metric space (X, d), we can define the Hausdorff distance for compact subsets of X:

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$S(\mathcal{A}^G)$ is homeomorphic to a closed subset of the Cantor space.
It is not itself a Cantor space because it has isolated points.

Positive cases

Groups for which projectively isolated subshifts are dense in $S(\mathcal{A}^G)$ ($|\mathcal{A}| > 1$):

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- Finitely generated free groups (Kwiatkowska, 2012)
- Free products of finite/cyclic groups (Doucha, 2024)

A criterion for the negative case

Theorem (C, arXiv:2601.03501)

Let G be a recursively presented group, and suppose that there exists a subshift of finite type $X \subset \mathcal{A}^G$ which is algorithmically complex in the following sense:

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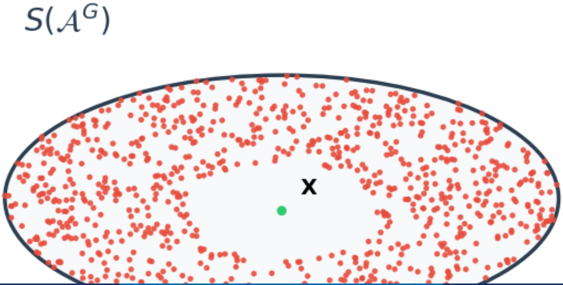

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This is currently the largest class of groups known to have no generic Cantor system.
One recovers previously known non-examples such as $\mathbb{Z}^2$ (Hochman, 2012), or products of three infinite groups (Doucha, 2024)

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For simplicity, assume in what follows that G has decidable word problem.

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Let $X_1, X_2 \dots$ in $S(\mathcal{A}^G)$ converge to a subshift of finite type X.
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Proposition

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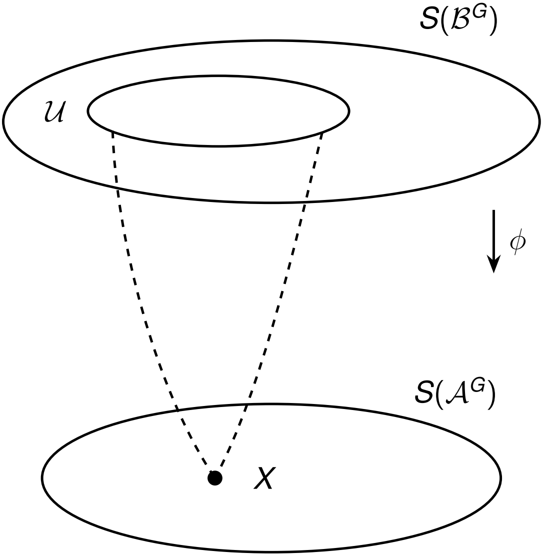
Putting these results together: if subshifts of finite type with no computable points happen to exist, then projectively isolated subshifts can not be dense (for some alphabet).

A subshift $X \in S(\mathcal{A}^G)$ is *projectively isolated* if we can find

- An alphabet $\mathcal{B}$
- A morphism of subshifts $\phi\colon \mathcal{B}^G \to \mathcal{A}^G$ (a CA)
- An open set $\mathcal{U}$ in $S(\mathcal{B}^G)$

such that $\phi(Y) = X$ for all $Y \in \mathcal{U}$.



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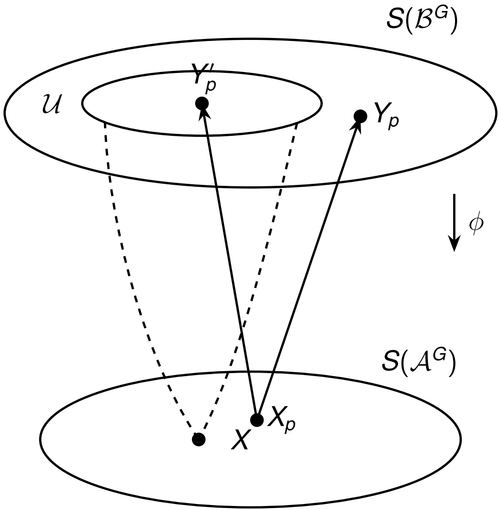
It is easy to see that $L(X)$ is Π^0_1, the nontrivial claim is that $L(X)$ is Σ^0_1.

Take $F \subset G$ and $p\colon F \to \mathcal{A}$ such that we want to know if $p \in L(X)$.

Algorithm:

1. Define $X_p = \{x \in X :$ p does not appear in $x\}$
2. For $\mathcal{U}$=ball around an SFT Y, define $Y_p = \{y \in Y : \phi(y) \in Y_p\} = \phi^{-1}(X_p) \cap Y$
3. Show that $p \in L(X) \iff Y_p \notin U$
4. Prove that $Y_p \notin \mathcal{U}$ is a Σ_1^0 problem.
5. $p \in L(X)$ is a Σ_1^0 problem.



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Given a group, one can also consider the space of *minimal* Cantor systems, and ask if there exists a generic minimal system.

Doucha, Tsankov, and Melleray proved that this question can also be reduced to the space of minimal subshifts.

Theorem (C and Doucha, in preparation)

Let G be a recursively presented group, and suppose that there exists an algorithmically complex subshift of finite type (no computable points). Then G does not admit a generic *minimal* Cantor system.

The general idea is the same. One can not prove that projectively isolated shifts have computable language, but one can still prove that they have Σ_1^0 language.

Thanks for your attention
and for the nice event