Medvedev degrees of subshifts

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Mostly based on a joint work with S. Barbieri

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Plan for the talk

1. Preliminaries
2. Introduction
3. Motivation
4. Medvedev degrees
5. The classification problem

Preliminaries

Simbolic dynamics, and a bit of computability.

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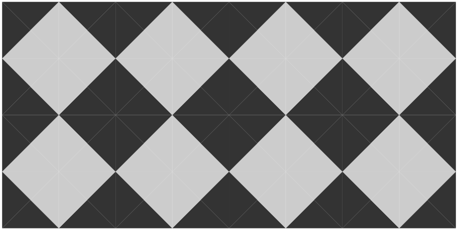

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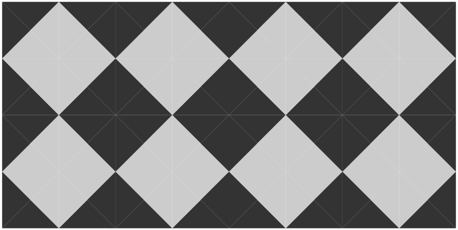
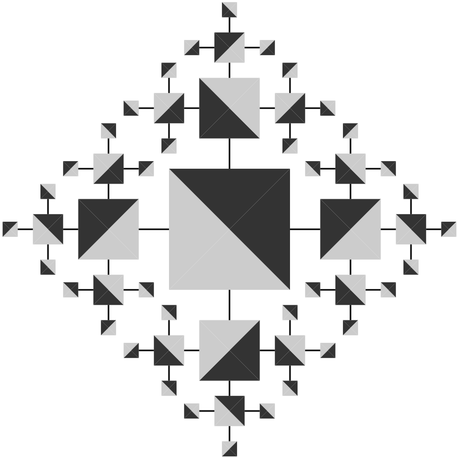


The set $X_\tau = \{x\colon \mathbb{Z}^2 \to \tau : x \text{ is a correct tiling } \}$ is a subshift of finite type.

Tilings of finitely generated groups

A correct tiling with τ on the free group with two generators:



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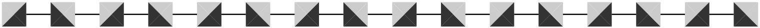

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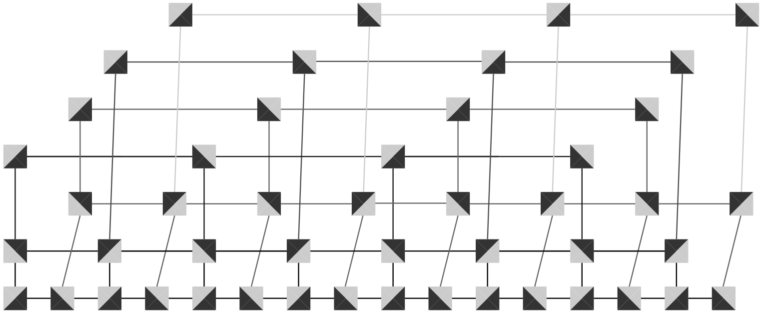

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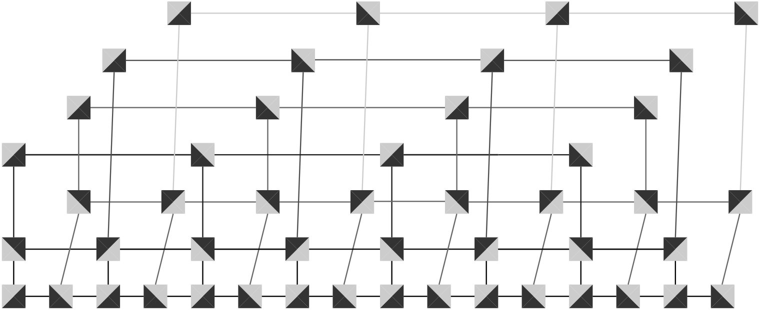

In a group with n generators, a tile would be n-dimensional cube with colored faces, or simply a tuple of colors of length $2n$.

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- **Shift action** = $G \curvearrowright A^G$, $gx(h) = x(g^{-1}h)$.
- **Subshift** = closed and shift-invariant subset of A^G.

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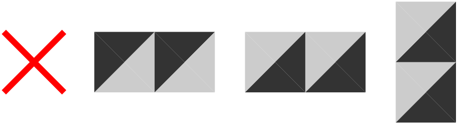

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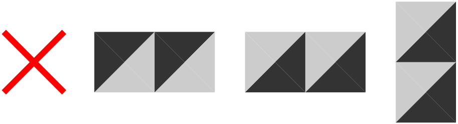

- Every subshifts of finite type is conjugate to X_τ for some finite tileset τ.

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Theorem (Curtis, Hedlund, Lyndon)

If ϕ is a morphism then there is $F \subset G$, finite alphabets A and B, and a "local function" $f\colon A^F \to B$ such that:

$$\phi(x)(g) = f((g^{-1}x)|_F)$$

Standing assumption

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Assuming decidable word problem is not needed for most results, but things are simpler.

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Non-Examples

The set of computable configurations is dense but countable.

Introduction

What is this thing?

The theorem of Hanf and Myers

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Remark

Intuitively, a puzzle with no computable solution.

A conjugacy invariant

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Remark

The property of having exclusively uncomputable elements is a conjugacy invariant.

Why is ”having only uncomputable points” an interesting property?

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An SFT with only uncomputable points is weakly aperiodic (no finite orbits).

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Proof.

Finite orbits are computable. □

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Corollary

Let X be an SFT with no computable points. Then an SFT contained in X is not minimal.

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A subshift is **sofic** when it is the topological factor of an SFT.

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Corollary

Let X be a sofic subshift with no computable points. Then any sofic subsystem of X is not minimal, nor a finite union of minimal subshifts.

The strong topological Rokhlin property

Definition

G has the strong topological Rokhlin property if the space of actions $G \curvearrowright \{0,1\}^{\mathbb{N}}$ contains a generic element.

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3. X $\mathbb{Z}^d$ for $d \geq 2$ (Hochman, 2012)
4. X Finitely generated nilpotent groups that are infinite and not virtually $\mathbb{Z}$ (Doucha, 2022) (Also other groups)

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Hochman actually proved that projectively isolated subshifts are not dense in $S(A^{\mathbb{Z}^2})$.

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Hochman constructed an SFT with this property, and used this to derive that $\mathbb{Z}^2$ does not have STRP.

Theorem (Hochman)

Let $\rho\colon Y \to Z$ be a factor map where Y is SFT, Z is the union of its minimal subsystems, and Z has no computable point. Let W be a subshift of finite type with a morphism $\pi\colon W \to Y$, so that W factors onto a subsystem of Y. Then for all ε there is a subshift $W_\varepsilon \subset W$ with $d_H(W, W_\varepsilon) \leq \varepsilon$ and $\pi(W) \neq \pi(W_\varepsilon)$.

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4. For well chosen n and $j \leq n$, the following works:

$$W_\varepsilon = (\pi\rho)^{-1}\Big(\bigcup_{\substack{1 \leq i \leq n \\ i \neq j}} Z_i \Big)$$

Medvedev degrees

What are Medvedev degrees?

The intuitive idea

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$$m(X)$$

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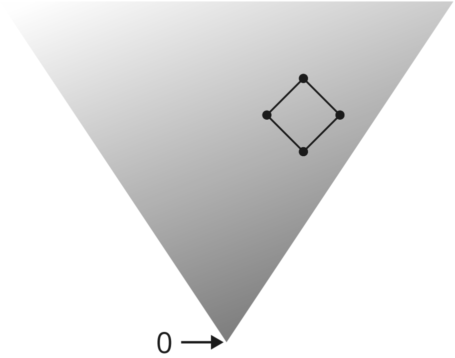
All the talk we have been looking at the property $m(X) \neq 0$.

The lattice of Medvedev degrees



Medvedev degrees have a partial order $\leq$, a minimal element 0, and operations of $\sup$ and $\inf$.

Medvedev degrees of subshifts

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4. $m(X \sqcup Y) = \inf\{m(X), m(Y)\}$.

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Let $P, Q \subset \{0,1\}^{\mathbb{N}}$.

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3. The Medvedev degree of P, written $m(P)$, is its equivalence class by $\equiv_M$.
4. $m(P) \leq m(Q) \iff P \leq_M Q$ defines a partial order on Medvedev degrees.

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Remark

Let X, Y be subshifts. Then $m(X) \geq m(Y)$ if and only if there is a computable function that maps X to Y.

The classification problem

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In what follows I will present some results around this question obtained with S.Barbieri.

⟨https://arxiv.org/abs/2406.12777⟩

The domino problem

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If G admits an SFT with $m(X) \neq 0$, then G has undecidable domino problem.

The domino problem for G is the algorithmic problem of determining whether an SFT presentation corresponds to an empty set.

Three questions about SFTs on groups

G admits SFTs with $m(X) \neq 0$?

G has undecidable domino problem?

G admits weakly aperiodic SFTs?

Three questions about SFTs on groups

G admits SFTs with $m(X) \neq 0$	$\Rightarrow$	G has undecidable domino problem
$\Downarrow$		
G admits weakly aperiodic SFTs		

Non-cases

Proposition (N.C., S.B)

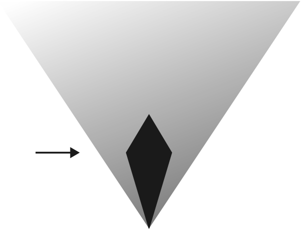
If G contains a finitely generated free group with finite index (virtually free), then $M_{SFT}(G) = \{0\}$.

These are the only known groups with $M_{SFT}(G) = \{0\}$.

An upper bound

Remark

If X is an SFT then $m(X)$ is a Π^0_1 degree.



Simpson's theorem

Theorem (S.Simpson 2012)

$$M_{SFT}(\mathbb{Z}^2) = \{ \text{ all } \Pi^0_1 \text{ degrees } \}$$

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We extended this classification to some other groups by studying how Medvedev degrees behave by group operations (subgroups, quotients, and others).

Translating properties

Proposition (N.C, S.B)

If G and H are commensurable, then $M_{SFT}(G) = M_{SFT}(H)$.

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There are also relations for inclusions, quotients, translation-like actions, etc.

Some new cases

Theorem (N.C, S.B)

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Some new cases

Theorem (N.C, S.B)

We have

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for the following groups:

1. Virtually polycyclic groups that are not virtually $\mathbb{Z}$
2. Branch groups with decidable word problem.
3. $G \times H$ where G and H are infinite and have decidable word problem.

Simulation results

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1. Baumslag-solitar groups $BS(1,n) = \langle a, b : ab = ba^n \rangle$
2. The Lamplighter group (Bartholdi, Salo 2024)
3. Self-simulable groups with decidable word problem (Barbieri, Sablik, Salo 2022)

Question

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Does every sofic subshift have an SFT extension with equal Medvedev degree?

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In $\mathbb{Z}^2$, does every sofic subshift have an SFT extension with equal topological entropy?

The end

Thanks

Thanks ! $(\ddot{\cup})$