The strong topological Rokhlin property and Medvedev degrees of SFTs

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Two actions $G \curvearrowright X$ and $G \curvearrowright Y$ are conjugate if there is an homeomorphism $\phi\colon X \to Y$ which intertwines the actions

$$\phi(gx) = g\phi(x)$$

The space of actions on the Cantor

Definition

A countable group G has the strong topological Rokhlin property (STRP) if the topological space of all continuous G-actions on the Cantor space

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- The space of actions is given the uniform topology.
- Here $T_n \to T$ if for every $g \in G$, $T_n^g(x)$ converges to $T(x)$ uniformly on $x \in \{0,1\}^{\mathbb{N}}$.
- The conjugacy class of T is $\{S : \text{ conjugate to T}\}$

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- ✗ Many groups (Doucha 2024)
- ✗ Bausmlag solitar groups, Lamplighter groups, some Branch groups, products of two infinite recursively presented groups, and some others (N.C-V)

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Theorem (Doucha 2024)

A finitely generated group G has the STRP if and only if *projectively isolated* subshifts are dense in the space of subshifts

$$S(A^G) = \{X \subset A^G : X \text{ subshift }\}$$

for every alphabet A with $|A| > 2$.

The space of subshifts

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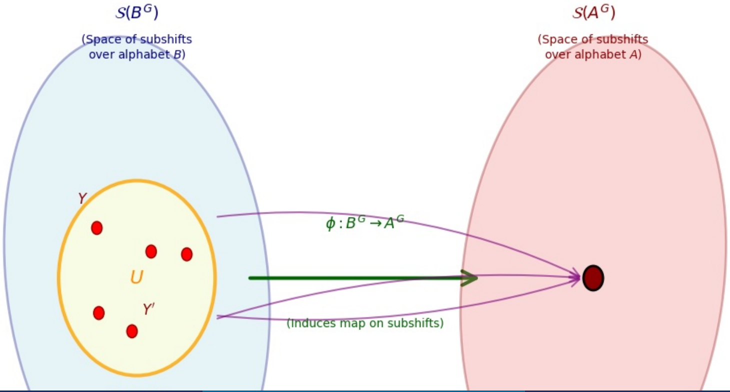
$$S(A^G) = \{X \subset A^G : \text{ subshift}\}$$
- $S(A^G)$ is homeomorphic to a closed subset of the Cantor space, but it has isolated points (e.g. any finite subshift).

Definition (Doucha, 2024)

A subshift $X \subset A^G$ is projectively isolated is there is an alphabet B, a continuous shift-equivariant map $\phi\colon B^G \to A^G$, and an open set U in $S(B^G)$ such that $\phi(Y) = X$ for all $Y \in U$.



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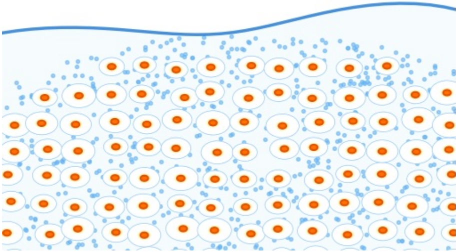

It is a Pełczyński space: a zero dimensional topological space with a dense collection of isolated points whose complement is a Cantor space.

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Since isolated subshifts are in particular projectively isolated, we recover Kechris and Rosendal's theorem that $\mathbb{Z}$ has the STRP.

Using Doucha's theorem, we can state an apparently simple criterion for the failure of the STRP.

Disproving the STRP

We have the following criterion to disprove the STRP.

Proposition

Let G be a finitely generated group. Suppose we have partially ordered set $(\mathfrak{R}, \leq_{\mathfrak{R}})$ with minimal element $0_{\mathfrak{R}}$, and a function

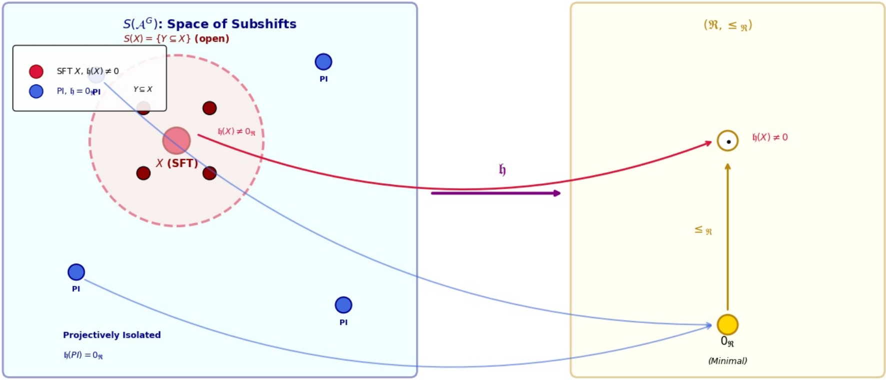
$$\mathfrak{h}\colon S(A^G) \to \mathfrak{R}$$

with these properties:

- $X \subset Y \Rightarrow \mathfrak{h}(X) \geq_{\mathfrak{R}} \mathfrak{h}(Y)$
- For every projectively isolated subshift X we have $\mathfrak{h}(X) = 0_{\mathfrak{R}}$.
- There is some SFT X with $\mathfrak{h}(X) \neq 0_{\mathfrak{R}}$.

Then G does not have the STRP.

Proof by AI-generated picture



Conclusion: PI subshifts cannot be dense ⇒ STRP fails

(by Doucha's Theorem)

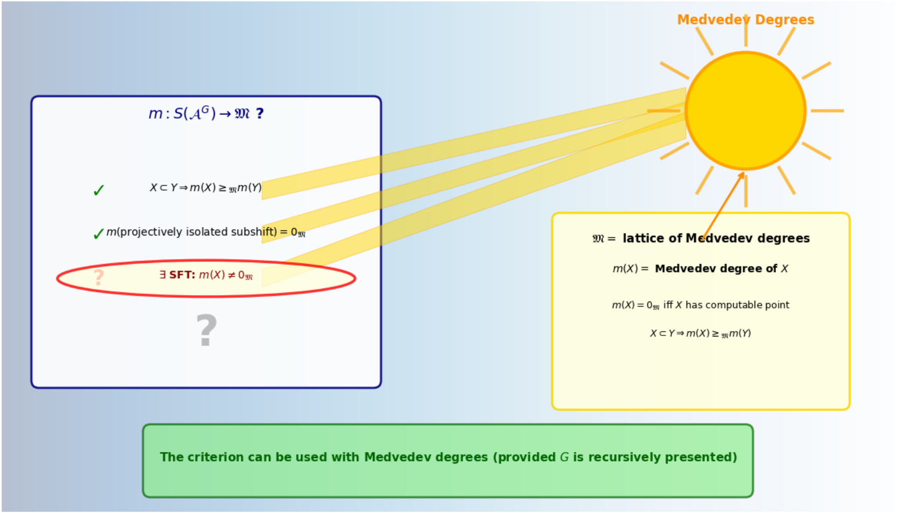
Question

To actually apply this criterion, we need to find a function $\mathfrak{h}$ with these properties.

Remark

We need that $\mathfrak{h}$ gets larger when we pass to a subsystem. Thus invariants such as entropy or variations of entropy won't work.

Medvedev degrees have the properties we need



Medvedev Degrees
$m : S(\mathcal{A}^G) \to \mathfrak{M}$?
✓ $X \subset Y \Rightarrow m(X) \geq_{\mathfrak{M}} m(Y)$
✓ m(projectively isolated subshift) $= 0_{\mathfrak{M}}$
? ∃ **SFT:** $m(X) \neq 0_{\mathfrak{M}}$
?
$\mathfrak{M} =$ **lattice of Medvedev degrees**
$m(X) =$ **Medvedev degree of** X
$m(X) = 0_{\mathfrak{M}}$ iff X has computable point
$X \subset Y \Rightarrow m(X) \geq_{\mathfrak{M}} m(Y)$
The criterion can be used with Medvedev degrees (provided G is recursively presented)

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- If $P \subset Q$, then finding elements in Q is easier than finding elements in P. Thus $m(P) \geq m(Q)$.

IF G is a finitely generated group then there is a natural way to computably identify A^G with a subset of $\{0,1\}^{\mathbb{N}}$, and then we can define $m(X)$ for a subshift $X \subset A^G$.

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The Curtis-Hedlund-Lyndon Theorem implies that continuous equivariant maps between subshifts are computable.

Basic properties of Medvedev degrees of subshifts

- Invariant for topological conjugacy
- X factors onto Y implies $m(X) \geq m(Y)$
- X embeds into Y implies $m(X) \geq m(Y)$
- $X \subset Y$ implies $m(X) \geq m(Y)$

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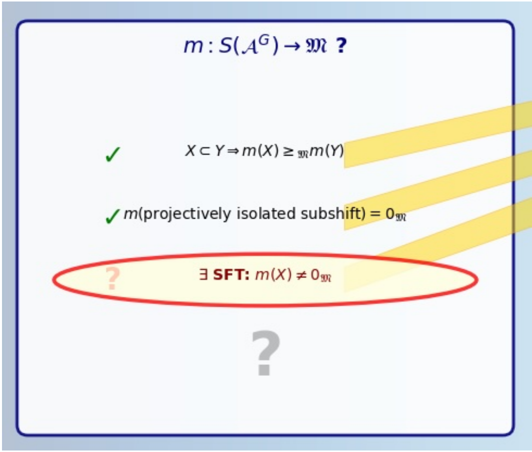

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✓more groups...

After we have a few groups admitting SFTs with nonzero Medvedev degree, we can obtain more.

Proposition (N.C and S.Barbieri, 2024)

Among finitely generated groups, the property "admitting an SFT with nonzero Medvedev degree" satisfies the following.

- It is transferred to supergroups.
- It is transferred to group extensions.
- It is a commensurability invariant.
- It a quasi-isometry invariant, provided the groups are finitely presented.

Proof ideas

The language of a G-subshift X is

$$L(X) = \{x|_F : x \in X, F \subset G \text{ finite}\}$$

Proposition (Folklore?)

A subshift with decidable language has zero Medvedev degree.

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This improves a previous result of me, Mathieu Sablik, and Alonso H. Nuñez, showing that isolated points in $S(A^G)$ are subshifts with decidable language provided G has decidable word problem (unpublished, but it appears on my phd thesis).

First half of the proof is for free

The co-language of a subshift $X \subset A^G$ is

$$L^c(X) := L(A^G) \setminus L(X)$$

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To prove that $L(X)$ is decidable, it is sufficient to prove that it is recursively enumerable.

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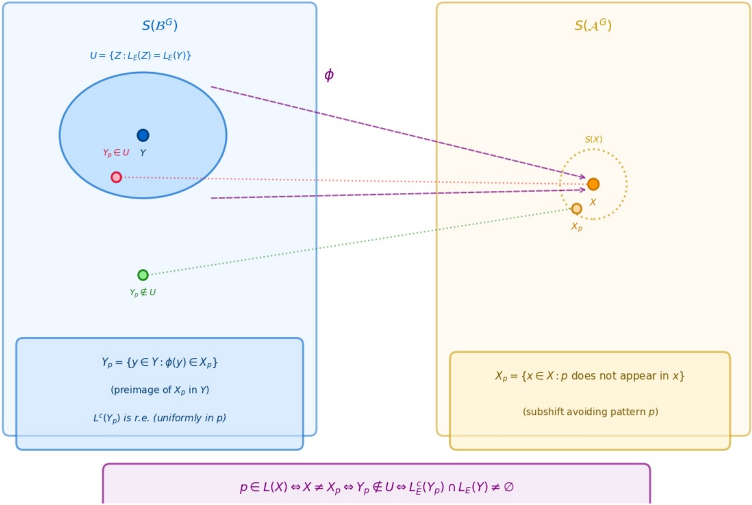

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Picture credits: Kimi AI - Cat credits: not known (open question)