

# SKEW PRODUCTS OVER IRRATIONAL ROTATIONS

SEMINAR TALK FOR DYNAMICS SEMINAR AT NICOLAUS COPERNICUS UNIVERSITY IN TORUŃ

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# INTRODUCTION

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In this talk all dynamical systems are measure-preserving and invertible.

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So no flows and no group actions (time =  $\mathbb{Z}$ ).

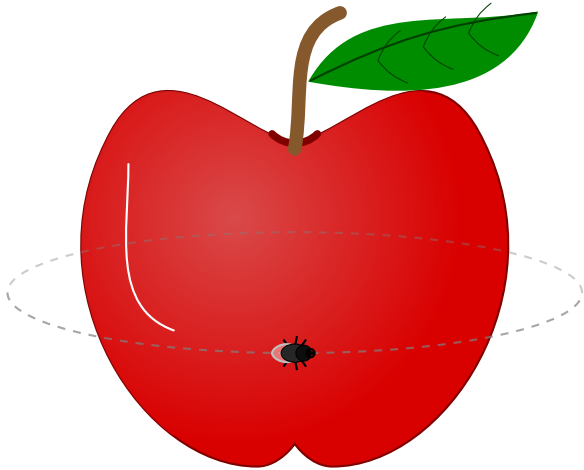
# INTRODUCTION

## KALIKOW'S $[T, T^{-1}]$ SYSTEM

The classic  $[T, T^{-1}]$  system is a skew product construction which formalizes the idea of applying a transformation  $T$  and its inverse  $T^{-1}$  randomly.

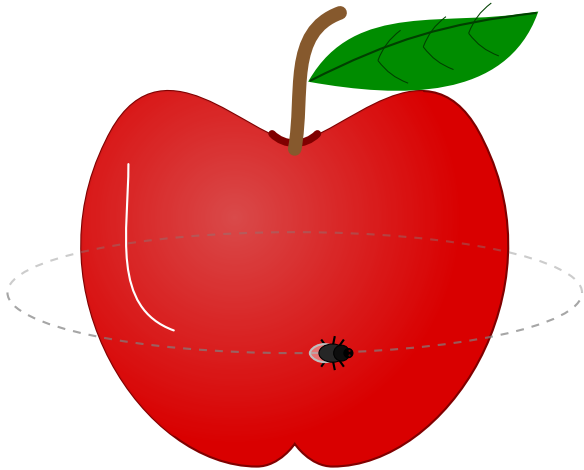
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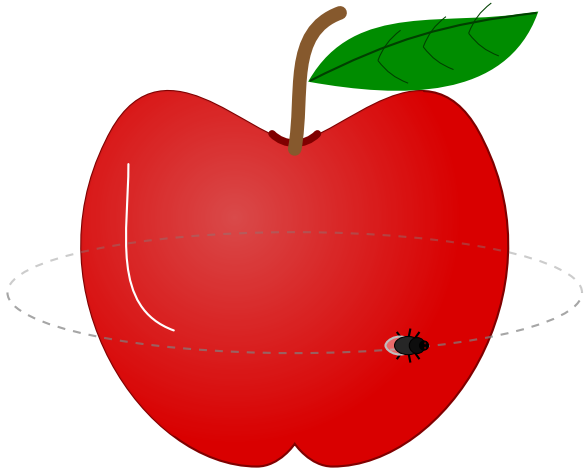
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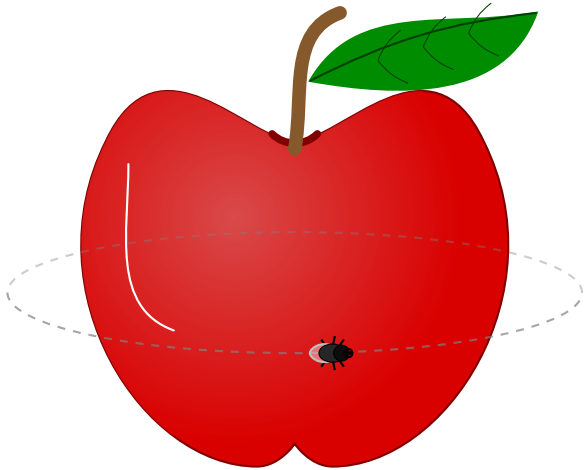
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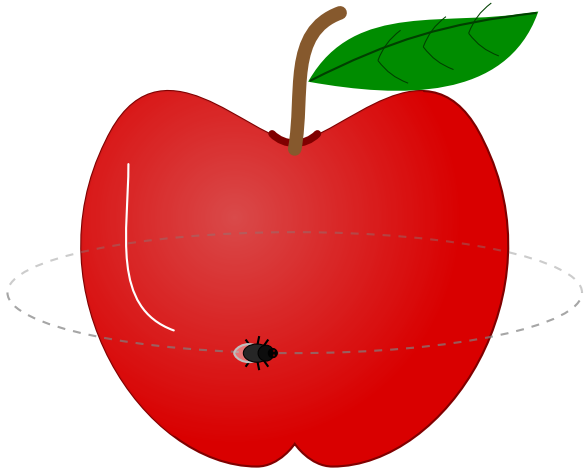
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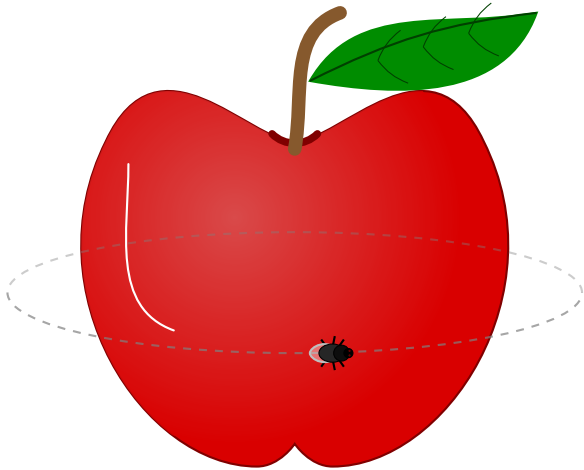
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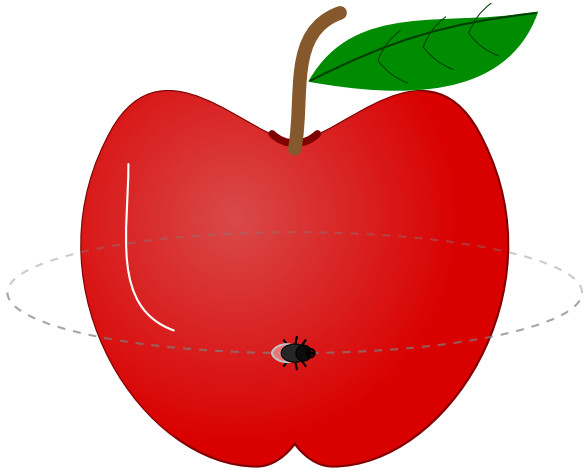
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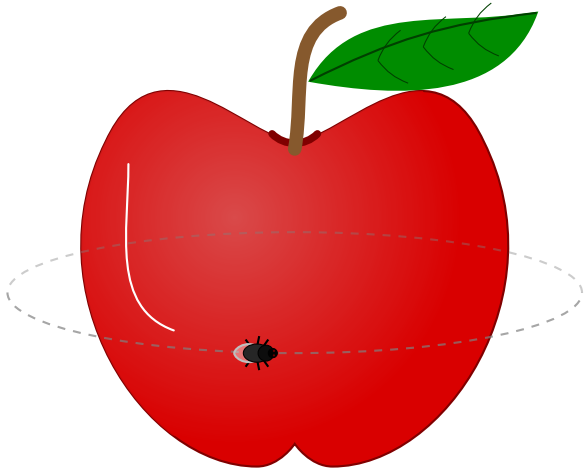
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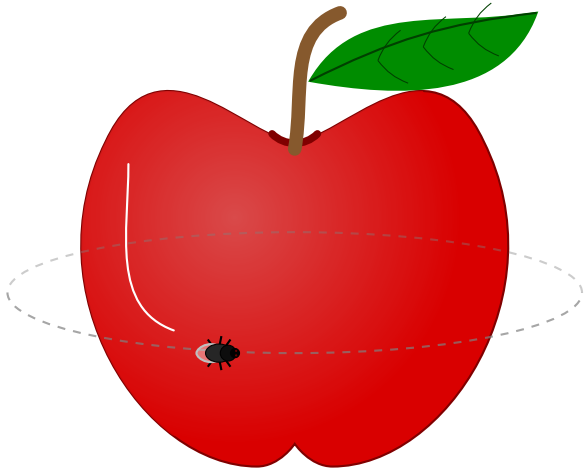
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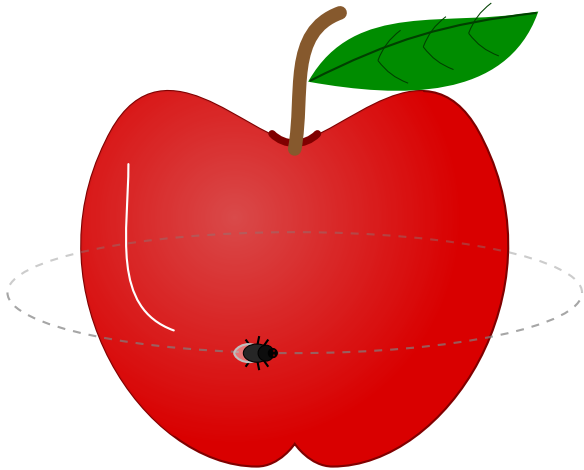
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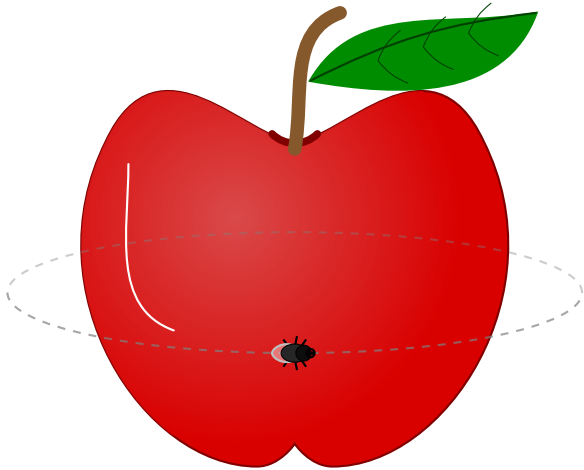
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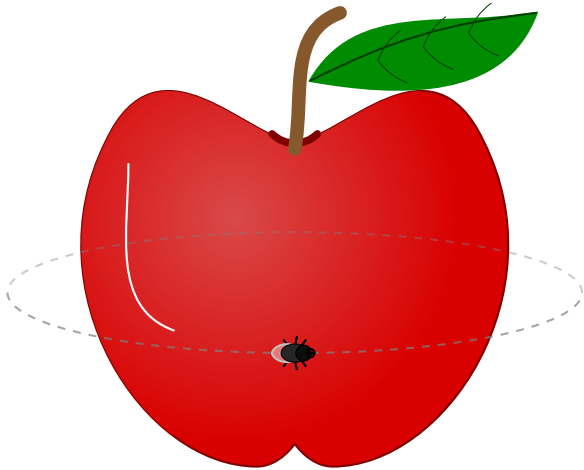
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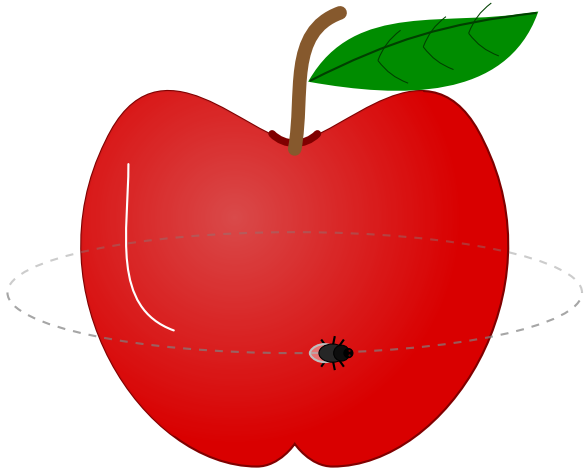
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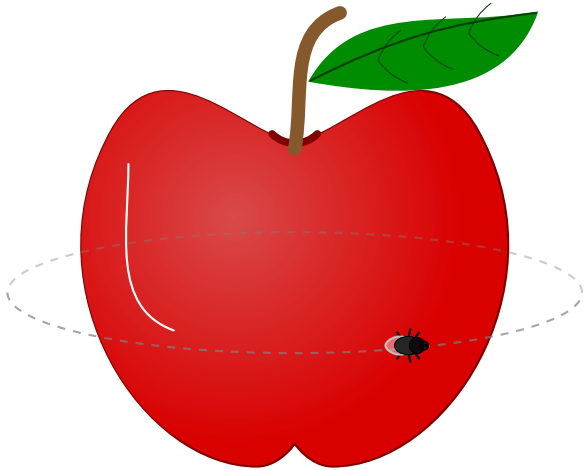
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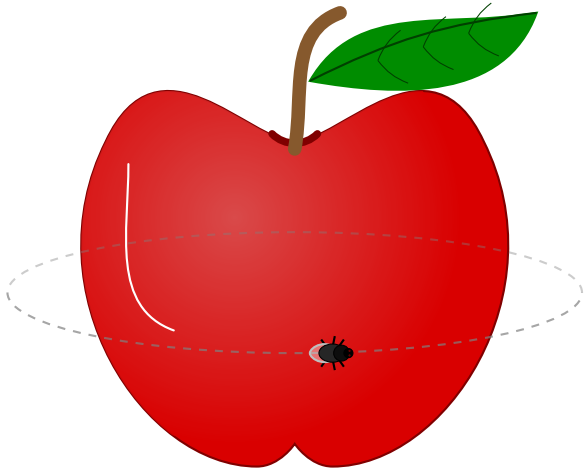
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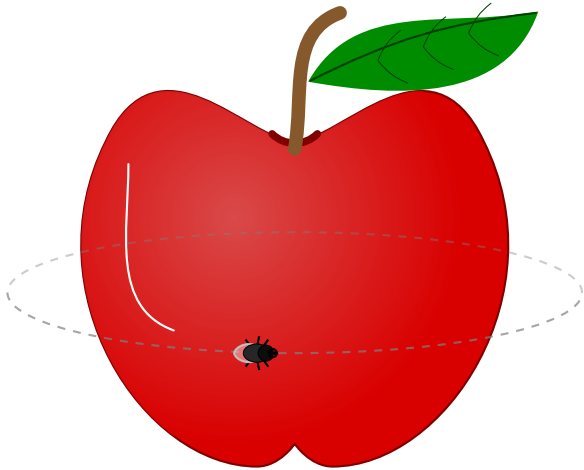
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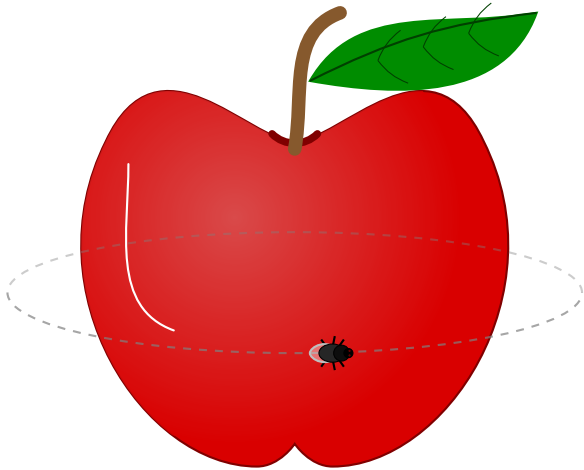
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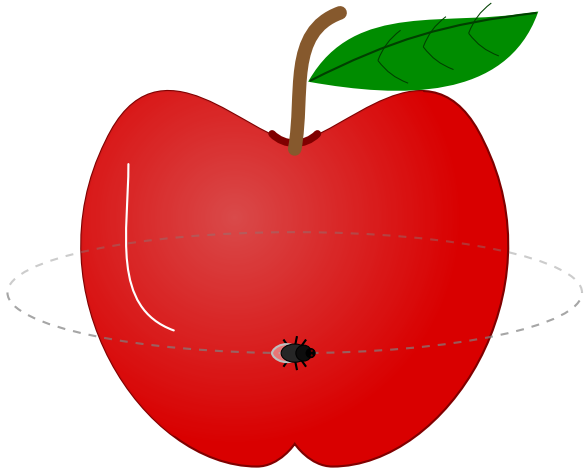
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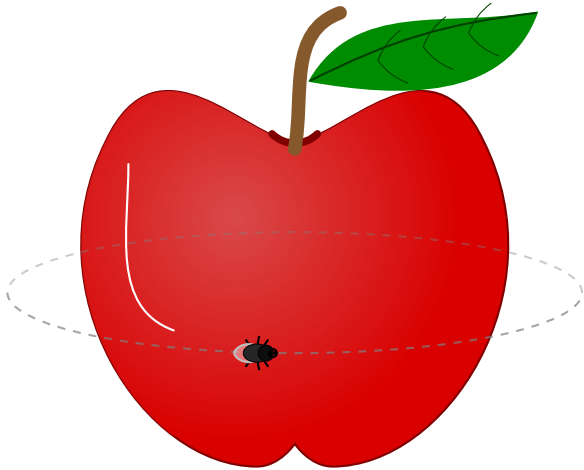
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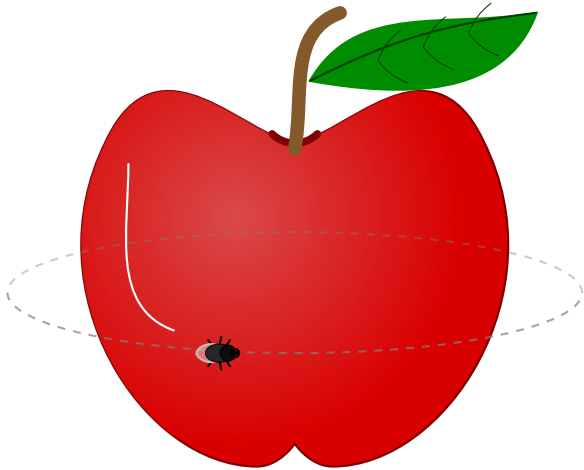
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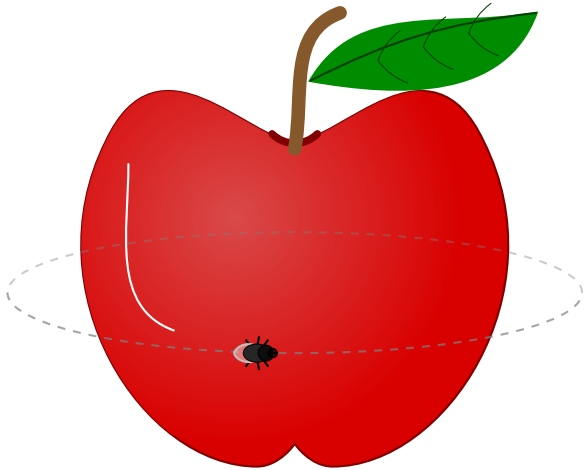
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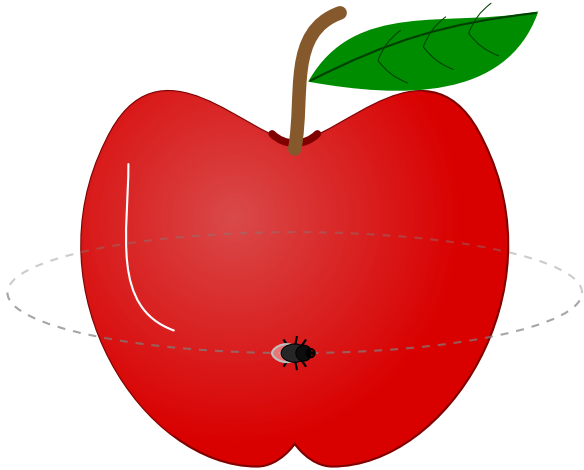
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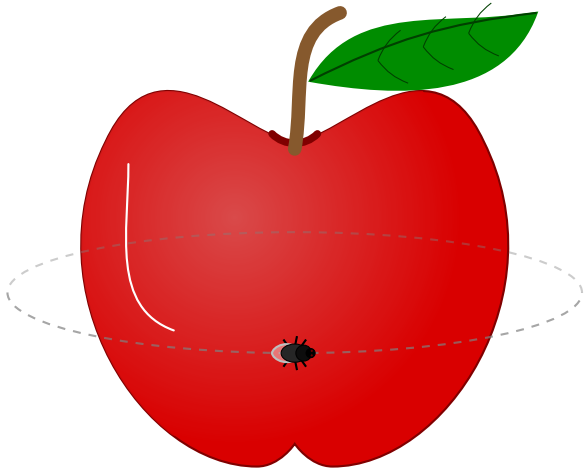
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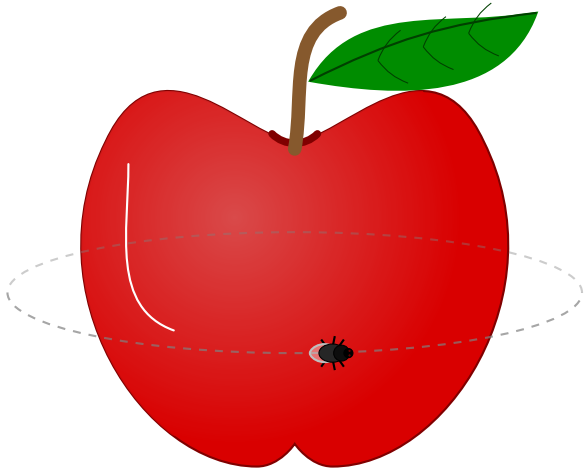
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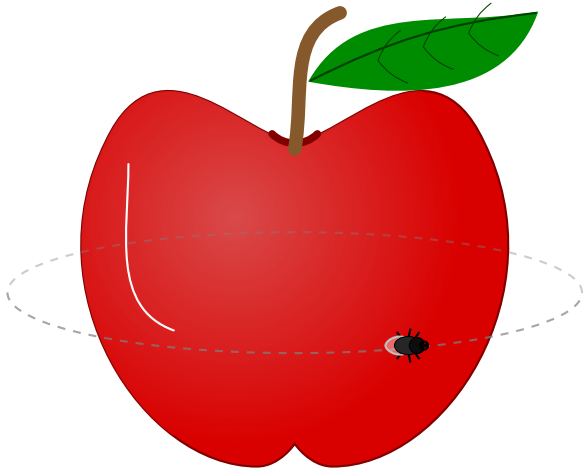
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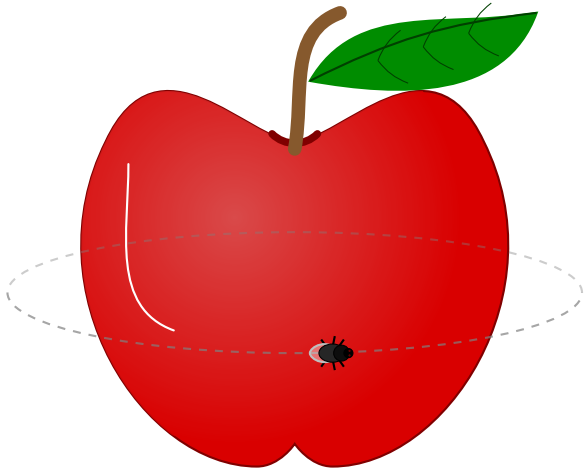
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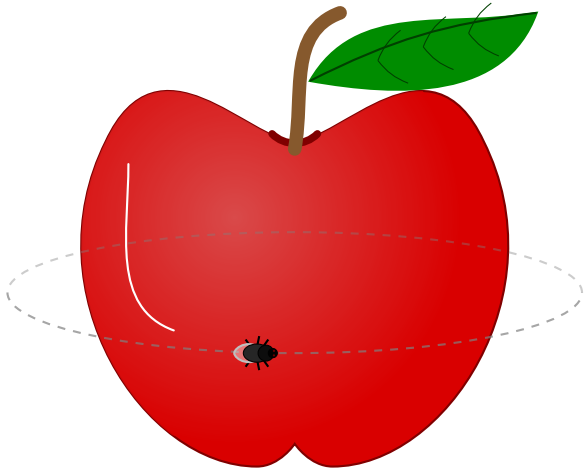
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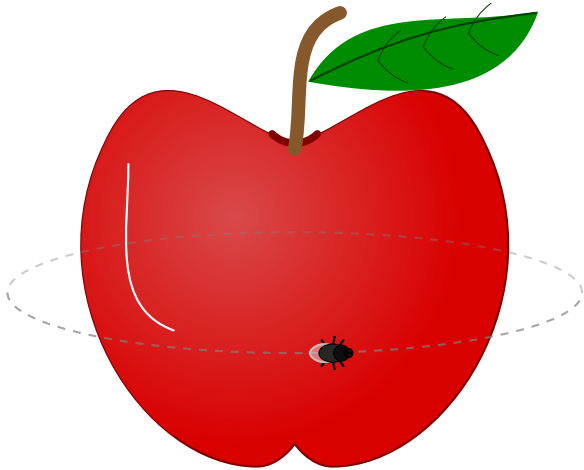
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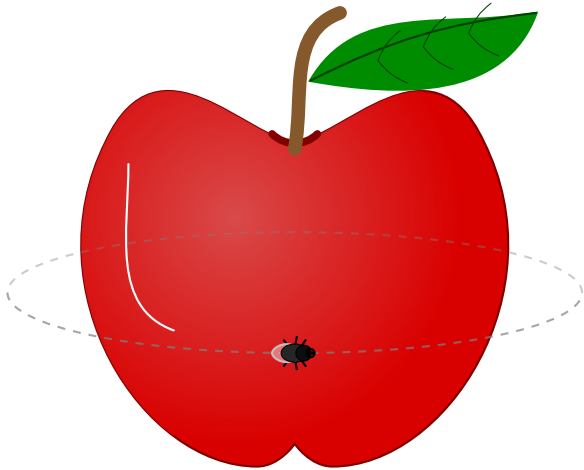
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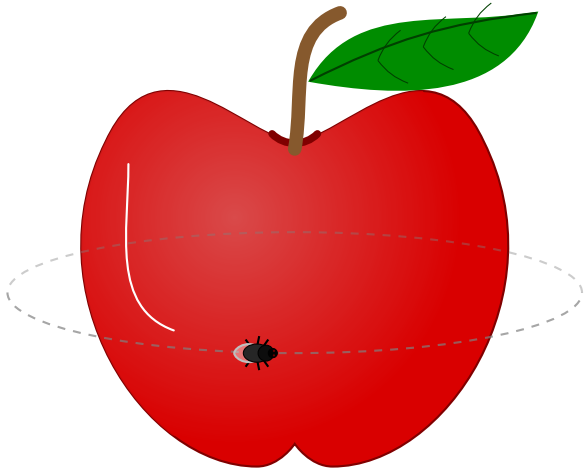
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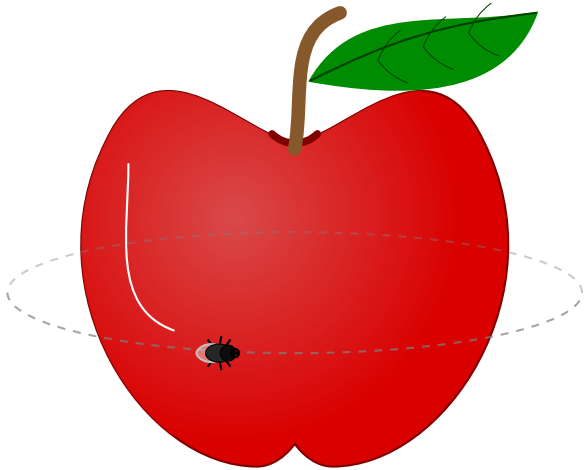
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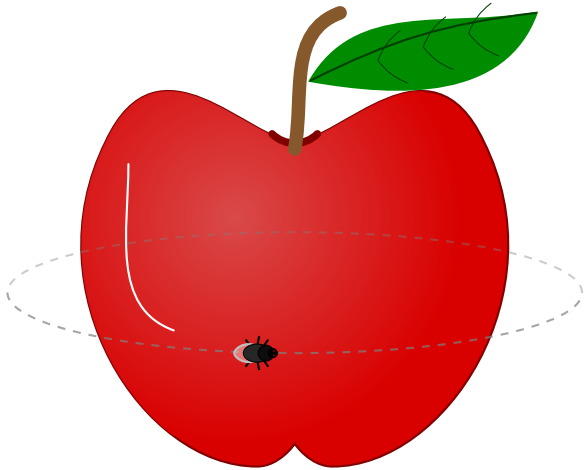
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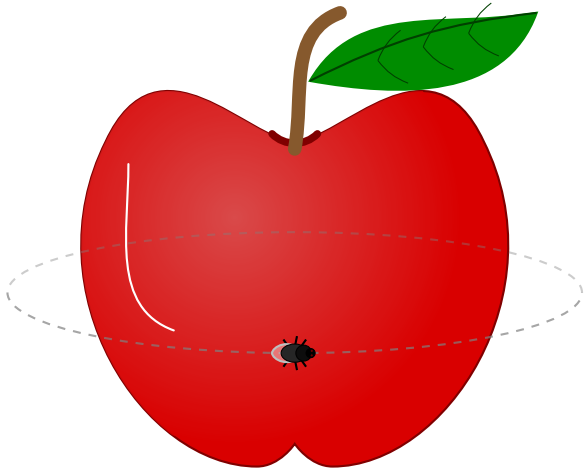
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## FUN FACT

### Remark

Some insects really move more or less randomly, or according a distribution called Levy walk, or reproduce more or less randomly.

- ▶ "Random Search Wired Into Animals May Help Them Hunt" in Quanta magazine.
- ▶ Popp, S. & Dornhaus, A. Ants combine systematic meandering and correlated random walks when searching for unknown resources. *iScience* 26, 105916 (2023).
- ▶ Curcio, L. et al. Double stochastic resonance in stink bug sexual communication. *Sci Rep* 16, 256 (2025)

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## KALIKOW'S $[T, T^{-1}]$ SYSTEM

### Ingredients

- ▶ Fiber: an automorphism  $T \curvearrowright (X, \mathcal{B}, \mu)$
- ▶ Base: the shift automorphism  $\sigma$  on  $(\{-1, 1\}^{\mathbb{Z}}, \mathcal{C}, \nu_{1/2, 1/2})$ .
- ▶ Selector (or cocycle):  $\tau: \{-1, 1\}^{\mathbb{Z}} \rightarrow \mathbb{Z}, y \mapsto \tau(y) = y(0)$

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### Construction

The  $[T, T^{-1}]$  transformation is the skew product

$$\sigma \rtimes_{\tau} T \curvearrowright (\{-1, 1\}^{\mathbb{Z}} \times X, \mathcal{C} \otimes \mathcal{B}, \nu_{1/2, 1/2} \otimes \mu)$$

$$(y, x) \mapsto (\sigma(y), T^{\tau(y)}(x))$$

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## WHY THIS INTERESTING

- ▶ Kalikow (1982) proved that if  $T$  has positive entropy then the  $[T, T^{-1}]$  system is Kolmogorov but not Bernoulli or loosely Bernoulli. It is the first natural or easily defined such example.

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## MAIN PROBLEM

### **Problem (A. Kanigowski)**

Are there similar results when the Bernoulli shift is replaced by a rotation?

# INTRODUCTION

## THE DETERMINISTIC VERSION OF THE $[T, T^{-1}]$ SYSTEM

### Ingredients

- ▶ Fiber: an automorphism  $T \curvearrowright (X, \mathcal{B}, \mu)$
- ▶ Base: an irrational rotation  $R_\alpha$  on  $\mathbb{T} = [0, 1)$  with the Borel sigma-algebra and the Lebesgue or Haar measure  $m$ .
- ▶ Selector (or cocycle):  $\tau: \mathbb{T} \rightarrow \mathbb{Z}, \theta \mapsto \mathbf{1}_{[0, 1/2)}(\theta) - \mathbf{1}_{[1/2, 1)}(\theta)$

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### Construction

Consider the skew product

$$R_\alpha \times_{\tau} T \curvearrowright (\mathbb{T} \times X, \mathcal{C} \otimes \mathcal{B}, m \times \mu)$$
$$(\theta, x) \mapsto (R_\alpha(\theta), T^{\tau(\theta)}(x))$$

## INTRODUCTION

### REMARKS

- ▶ The concrete problem is to distinguish  $R_\alpha \rtimes_\tau T_1$  and  $R_\alpha \rtimes_\tau T_2$  for different  $T_1$  and  $T_2$ .

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- ▶ Entropy gives no information because we always have  $h_{m \times \mu}(R_\alpha \rtimes_\tau T) = 0$ .
- ▶ The powers of  $R_\alpha \rtimes_\tau T$  are determined to the ergodic sums of  $\tau = \mathbf{1}_{[0,1/2)} - \mathbf{1}_{[1/2,1)}$

$$\sum_{i=1}^{n-1} \tau \circ R_\alpha^i$$

in the sense that

$$(R_\alpha \rtimes_\tau T)^n(\theta, x) = (R_\alpha^n(\theta), T^{\sum_{i=1}^{n-1} \tau(R_\alpha^i(\theta))}(x))$$

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- ▶ The sequence  $\sum_{i=1}^{n-1} \tau \circ R_\alpha^i$  of measurable functions  $\mathbb{T} \rightarrow \mathbb{Z}$  is sometimes called deterministic random walk and has nice properties.

# INTRODUCTION

## VISUALIZATION

Let  $T: [0, 1] \rightarrow [0, 1], x \mapsto \sqrt{x}$



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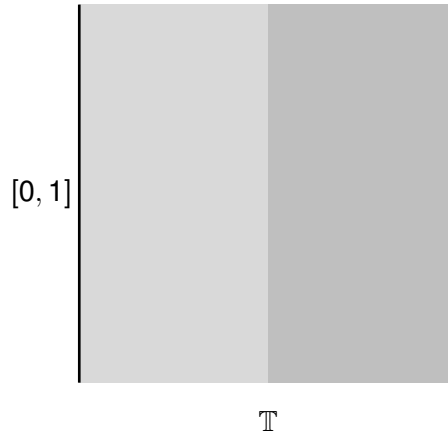
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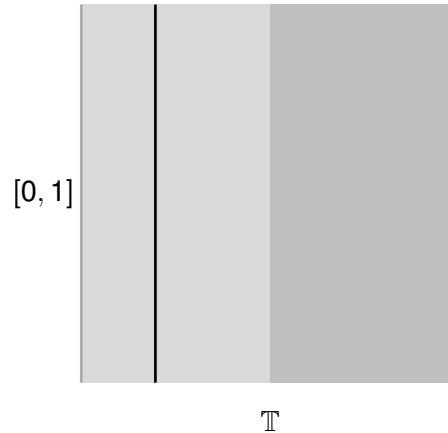
Visualization of  $R_\alpha \times_\tau T \curvearrowright [0, 1]^2$  with  $T(x) = \sqrt{x}$  on  $[0, 1]$  ( $\alpha = 1/\sqrt{26}$ )



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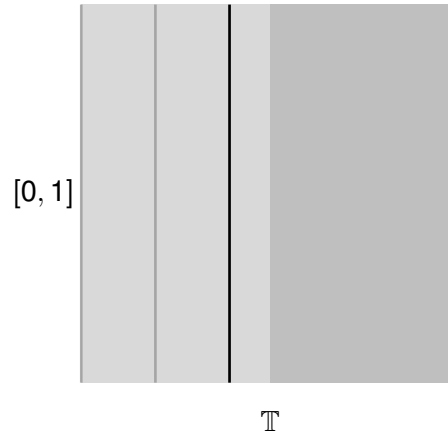
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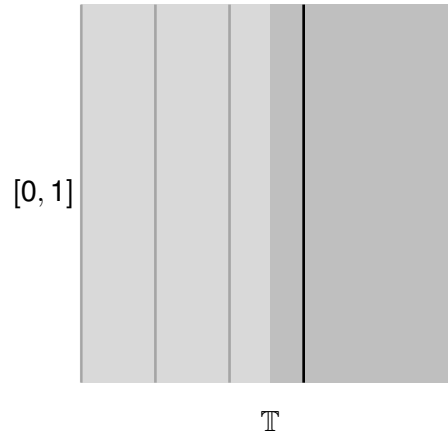
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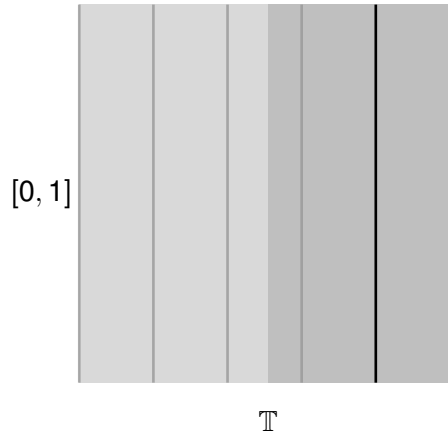
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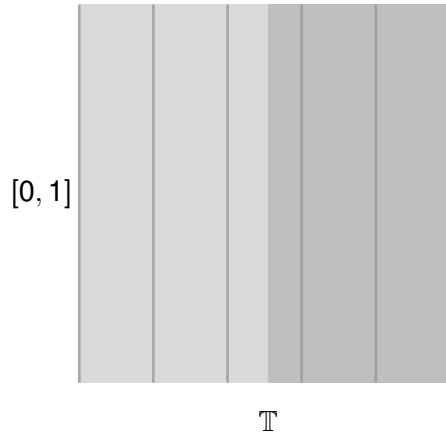
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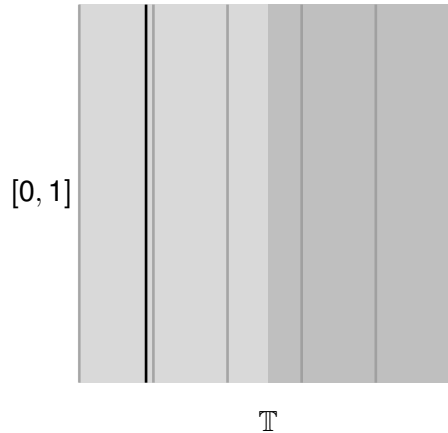
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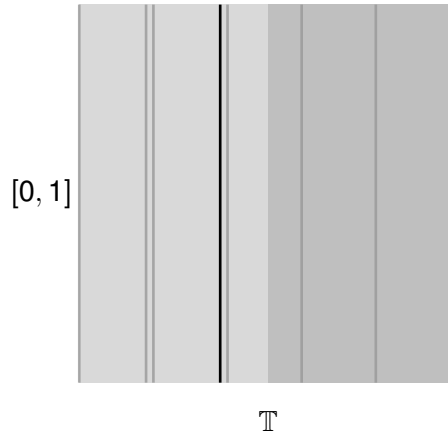
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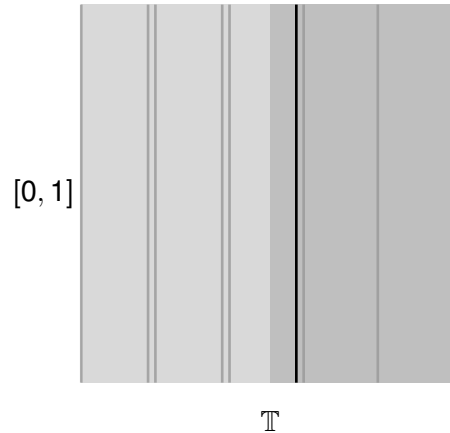
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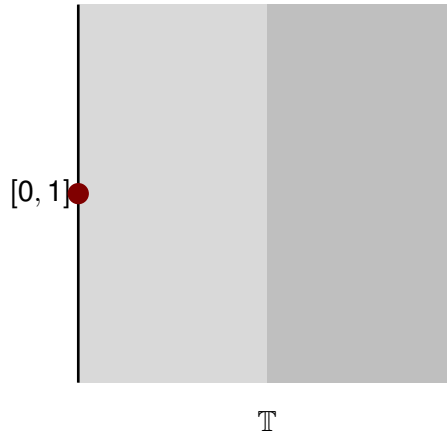
Visualization of  $R_\alpha \times_\tau T \curvearrowright [0, 1]^2$  with  $T(x) = \sqrt{x}$  on  $[0, 1]$  ( $\alpha = 1/\sqrt{26}$ )



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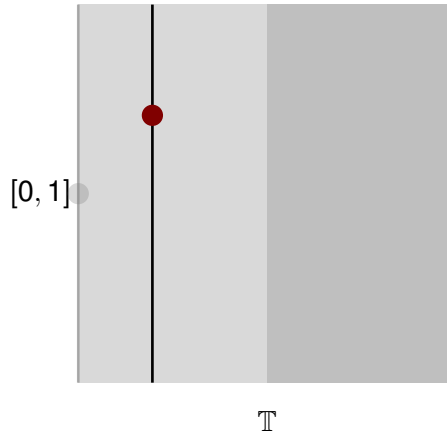
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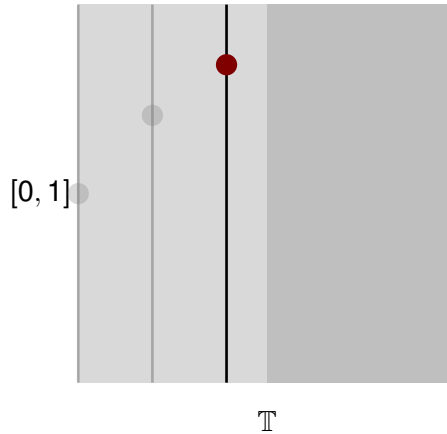
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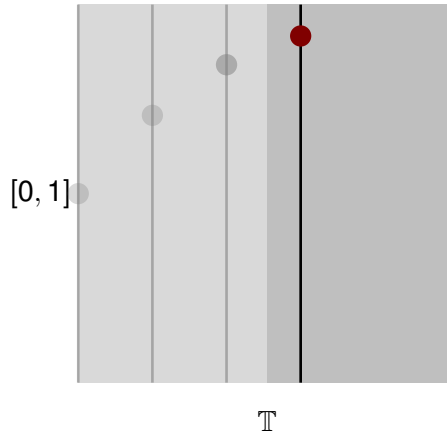
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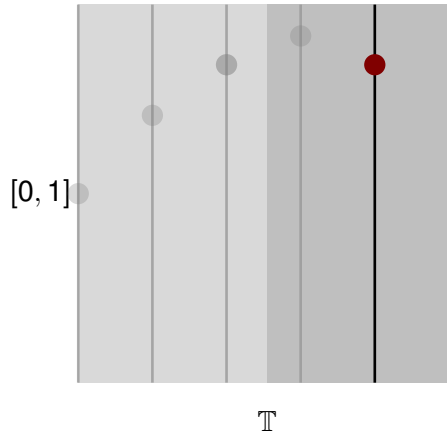
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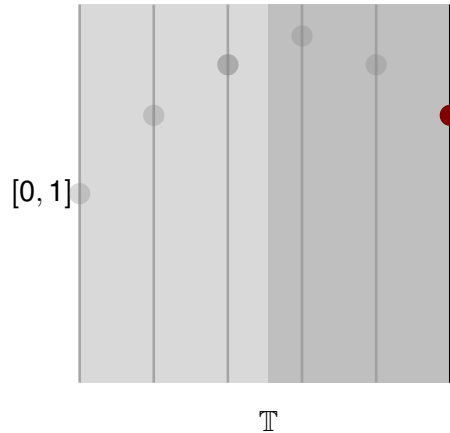
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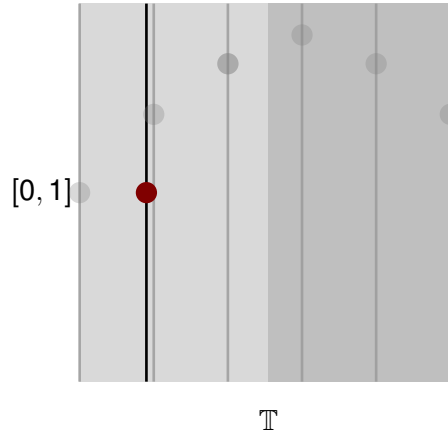
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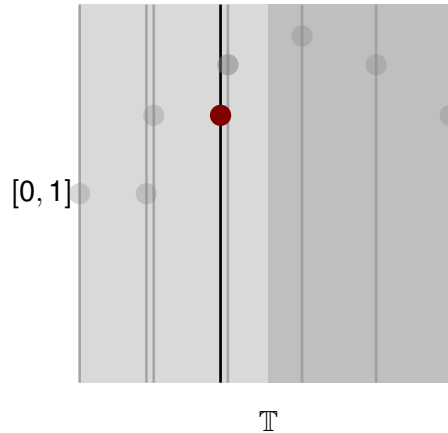
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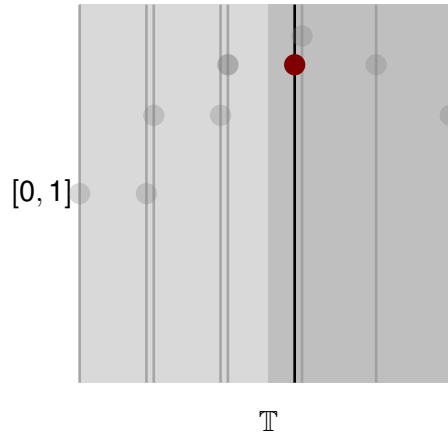
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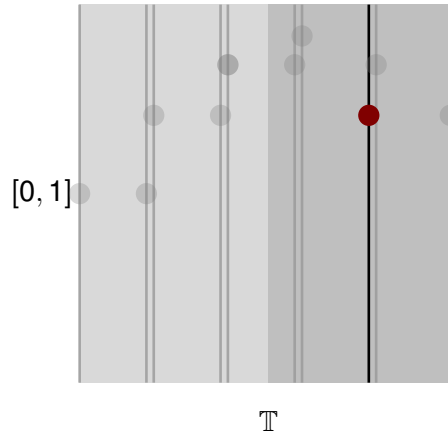
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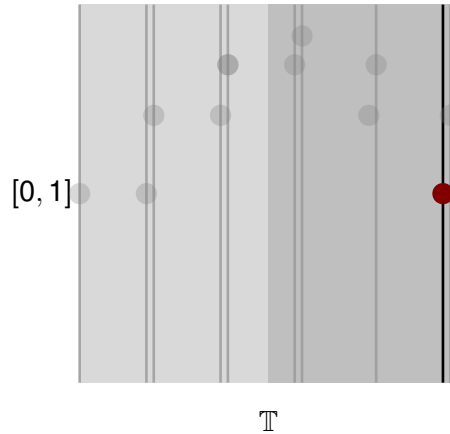
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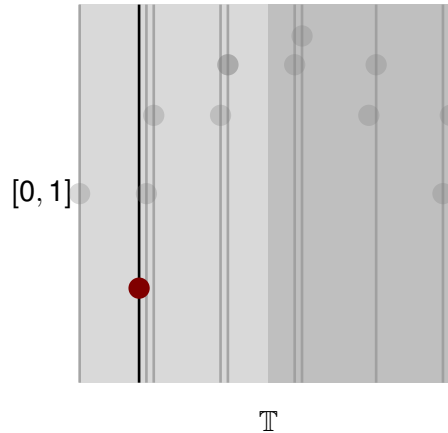
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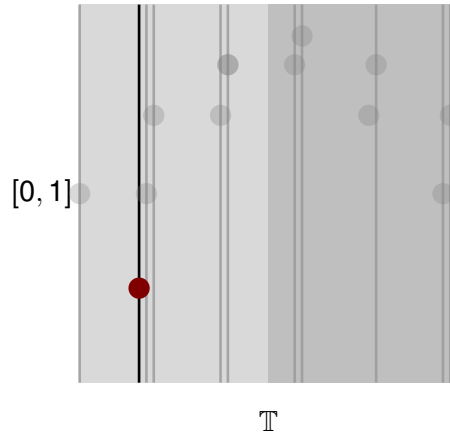
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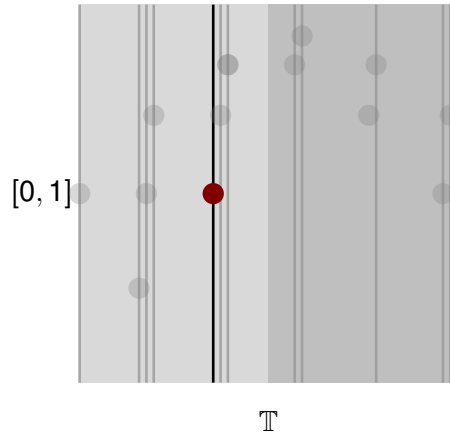
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Under some conditions on the rotation angle  $\alpha$  (satisfied for a set with full measure) we have

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- ▶ ✓ results analogous to the existing ones for  $[T, T^{-1}]$  systems (proved by Austin, Hoffman, Hecklen, and Rudolf, and then generalized by Ball);
- ▶ ✓ also stronger results.

# RESULTS

## THE CONDITION ON THE ANGLE

### The diophantine condition

We say that  $\alpha$  is good if for every  $k \in \mathbb{N}$  we can find  $p, q$  coprime with  $q$  odd and

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$$|\alpha - p/q| < \frac{1}{kq^2}$$

For instance, if  $\alpha = [0; a_1, a_2, \dots]$  and  $a_n \rightarrow \infty$ , then  $\alpha$  is good.

# RESULTS

## THE SKEW PRODUCT REMEMBERS THE ENTROPY OF THE FIBER

### Theorem 1

Let  $\alpha$  be good. There is an invariant for isomorphism  $I$  depending on  $\alpha$  such that for any automorphism  $T \curvearrowright (X, \mathcal{B}, \mu)$  we have

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Note: this is a literal analogue of the existing results for classic  $[T, T^{-1}]$  systems.

# RESULTS

## THE SKEW PRODUCT REMEMBERS THE SLOW ENTROPY OF THE FIBER

### Theorem 2

Let  $\alpha$  be good and let  $\mathbf{a}$  be a scale for slow entropy (will define later). There is an invariant for isomorphism  $I$  depending on  $\alpha$  such that for any automorphism  $T \curvearrowright (X, \mathcal{B}, \mu)$  we have

$$I(R_\alpha \times_\tau T) = \text{ent}_\mu^{\mathbf{a}}(T)$$

## RESULTS

### THE SKEW PRODUCT REMEMBERS THE FLIP ISOMORPHISM CLASS OF THE FIBER

#### Theorem 3

*Let  $\alpha$  be good and  $T_1, T_2 \curvearrowright (X, \mathcal{B}, \mu)$  be weakly mixing automorphisms. Then  $R_\alpha \rtimes_\tau T_1$  is isomorphic to  $R_\alpha \rtimes_\tau T_2$  if and only if  $T_1$  is isomorphic to  $T_2$  or the inverse of  $T_2$ .*

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This is surprising when compared to the following.

#### Example (due to M. Lemańczyk, see [arxiv:2509.09003](https://arxiv.org/abs/2509.09003))

There exist non-isomorphic weakly mixing automorphisms  $T_1$  and  $T_2$  such that  $R_\alpha \times T_1$  is isomorphic to  $R_\alpha \times T_2$ .

## PROOF IDEAS FOR THE ISOMORPHISM THEOREM

We review some ideas in the proof of the isomorphism theorem - this will lead us to the proofs of the entropy theorems.

## PROOF IDEAS FOR THE ISOMORPHISM THEOREM

Let  $T_1$  and  $T_2$  be weakly mixing automorphisms of a common Borel probability space  $(X, \mathcal{B}, \mu)$ . Assume that  $R_\alpha \times_{\tau} T_1$  is isomorphic to  $R_\alpha \times_{\tau} T_2$  via

$$\Phi: \mathbb{T} \times X \rightarrow \mathbb{T} \times X$$

$$(\theta, x) \rightarrow (\phi(\theta, x), \psi(\theta, x))$$

We write  $\psi_\theta(x) = \psi(\theta, x)$ .

### Goal

$T_1$  is isomorphic to  $T_2$  or  $T_2^{-1}$

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4. Find  $\theta^*$  with  $\psi_{\theta^*} \circ T_1 = T_2 \circ \psi_{\theta^*}$  or  $\psi_{\theta^*} \circ T_1 = T_2^{-1} \circ \psi_{\theta^*}$

# PROOF IDEAS FOR THE ISOMORPHISM THEOREM

## STEP 4

### Proposition

Suppose that  $p, q \in \mathbb{N}$  are coprime,  $q$  odd, and  $|\alpha - p/q| < 1/(5q^2)$ . Then for every  $\theta \in \mathbb{T}$  we have  $\sum_{i=0}^{q-1} \tau(\theta +_{\mathbb{T}} i\alpha) \in \{-1, 1\}$

### Definition

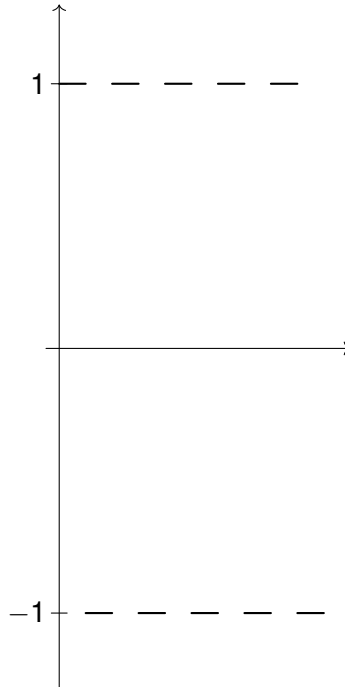
For such  $q$  we define

$$A_+(q) = \left\{ \theta \in \mathbb{T} : \sum_{i=0}^{q-1} \tau(\theta +_{\mathbb{T}} i\alpha) = 1 \right\}$$
$$A_-(q) = \left\{ \theta \in \mathbb{T} : \sum_{i=0}^{q-1} \tau(\theta +_{\mathbb{T}} i\alpha) = -1 \right\}$$

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## STEP 4

For  $q$  odd and  $|\alpha - p/q| < 1/(5q^2)$ ,  $\theta \rightarrow \sum_{i=0}^{q-1} \tau(\theta + \mathbb{T} i\alpha)$  looks like



## PROOF IDEAS FOR THE ISOMORPHISM THEOREM

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Fix an increasing sequence  $q_n$  of odd numbers with  $|\alpha - p_n/q_n| < 1/(n^2 q_n^2)$ .

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$$\begin{aligned} 1/2 &= \lim_{n \rightarrow \infty} m(A_+(q_n) -_{\mathbb{T}} \mathbf{c}) = \lim_{n \rightarrow \infty} m((A_+(q_n) -_{\mathbb{T}} \mathbf{c}) \cap (A_+(q_n) \cup A_-(q_n))) \\ &\leq \limsup_{n \rightarrow \infty} m((A_+(q_n) -_{\mathbb{T}} \mathbf{c}) \cap A_-(q_n)) + \limsup_{n \rightarrow \infty} m((A_+(q_n) -_{\mathbb{T}} \mathbf{c}) \cap A_+(q_n)) \end{aligned}$$

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Some of the limsups must be positive. We assume this for the second limsup and prove  $T_1 \equiv T_2$ . In the other case the same argument shows  $T_1 \equiv T_2^{-1}$ .

## PROOF IDEAS FOR THE ISOMORPHISM THEOREM

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Since  $\Phi$  is an isomorphism, for any  $\ell \geq 1$  we have  $\Phi((R_\alpha \rtimes_\tau T_1)^\ell(\theta, x)) = (R_\alpha \rtimes_\tau T_2)^\ell(\Phi(\theta, x))$ . Looking at the second coordinate we find

$$\psi_{\theta+\ell\alpha} \circ T_1^{\sum_{i=0}^{\ell-1} \tau(\theta+i\alpha)} = T_2^{\sum_{i=0}^{\ell-1} \tau(\theta+c+i\alpha)} \circ \psi_\theta$$

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Passing to a subsequence if needed, we can assume that  $\theta_n$  converges to a limit  $\theta^*$ . If  $\theta \rightarrow \psi_\theta$  was continuous then we would have

$$\psi_{\theta^*} \circ T_1 = \lim_{n \rightarrow \infty} \psi_{\theta_n+q_n\alpha} \circ T_1 = \lim_{n \rightarrow \infty} T_2 \circ \psi_{\theta_n} = T_2 \circ \psi_{\theta^*}$$

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For every  $n \geq N$  we pick  $\theta_n \in (A_+(q_n) -_{\mathbb{T}} c) \cap A_+(q_n)$  and apply the previous relation to  $\ell = q_n$  to find

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Passing to a subsequence if needed, we can assume that  $\theta_n$  converges to a limit  $\theta^*$ . If  $\theta \rightarrow \psi_\theta$  was continuous then we would have

$$\psi_{\theta^*} \circ T_1 = \lim_{n \rightarrow \infty} \psi_{\theta_n+q_n\alpha} \circ T_1 = \lim_{n \rightarrow \infty} T_2 \circ \psi_{\theta_n} = T_2 \circ \psi_{\theta^*}$$

It is not continuous, but we can apply Lusin's Theorem to obtain that  $\theta \rightarrow \psi_\theta$  is continuous on a set with arbitrarily large measure.

## SOME REMARKS ABOUT SIGMA-ALGEBRAS

- ▶ In the previous proof we considered an isomorphism

$$\Phi: \mathbb{T} \times \mathcal{X} \rightarrow \mathbb{T} \times \mathcal{X}$$

$$(\theta, \mathbf{x}) \rightarrow (\phi(\theta, \mathbf{x}), \psi(\theta, \mathbf{x}))$$

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- ▶ We started with the claim that  $\phi$  depends only on  $\theta$  a.e.
- ▶ But, how do we get there?.

## SOME REMARKS ABOUT SIGMA-ALGEBRAS

### Proposition

Let  $(Y, \mathcal{C}, \nu)$  and  $(X, \mathcal{B}, \mu)$  be standard Borel probability spaces, and consider their product  $(Y \times X, \mathcal{C} \otimes \mathcal{B}, \nu \times \mu)$ . Consider an isomorphism

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### Proof.

Doob's Lemma from probability.



### Remark

Our problem gets reduced to proving that isomorphisms are measurable with respect to certain sub-sigma-algebra.

# SOME REMARKS ABOUT SIGMA-ALGEBRAS

## THE PINSKER $\sigma$ -ALGEBRA

### Definition

Let  $T$  be an automorphism of a standard Borel probability space  $(X, \mathcal{B}, \mu)$ . The Pinsker  $\sigma$ -algebra of  $T$  is

$$\Pi(T) = \{P \in \mathcal{B} : h_\mu(T, \{P, X \setminus P\}) = 0\}$$

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An isomorphism between two systems  $T_1$  and  $T_2$  is necessarily measurable with respect to the Pinsker  $\sigma$ -algebras.

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#### Remark

In order to get to step 1 in the previous proof, we use an analogous of the Pinsker  $\sigma$ -algebra but with a variation of slow entropy.

# A VARIATION OF SLOW ENTROPY

## SOME DEFINITIONS

Let  $T$  be an automorphism of a standard Borel probability space  $(X, \mathcal{B}, \mu)$  and let  $\xi$  be a finite measurable partition.

### Dinamically generated Hamming pseudo-distances

Given  $F \subset \mathbb{Z}$  finite,

$$d_{\xi, F}^{H, T}(x, y) = \frac{1}{|F|} |\{i \in F : \xi(T^i(x)) \neq \xi(T^i(y))\}|$$

### Covering numbers

Given  $F \subset \mathbb{Z}$  finite,  $\epsilon > 0$ ,  $\delta > 0$ , define

$S_{\xi}^H(T, F, \epsilon, \delta) =$  minimal number of  $\epsilon$ -balls in  $d_{\xi, F}^{H, T}$  needed to cover measure  $\mu$  larger than  $1 - \delta$

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### Scales

A scale  $\mathbf{a} = \{a_n(t)\}_{n \in \mathbb{N}, t > 0}$  is a sequence of non-decreasing functions  $a_n: (0, \infty) \rightarrow (0, \infty)$  such that  $\lim_{n \rightarrow \infty} a_n(t) = \infty$  for all  $t > 0$ .

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### The set of iterates

Let  $\mathcal{F} = (F_n)_{n \in \mathbb{N}}$  be a sequence of finite subsets of  $\mathbb{Z}$  with  $\lim_{n \rightarrow \infty} |F_n| = \infty$ .

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Given a finite partition  $\xi$  of  $X$  we define

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### The invariant

To obtain an invariant for isomorphism we take supremum over all finite partitions

$$\text{ent}_{\mu}^{\mathcal{F}, \mathbf{a}}(T) = \sup_{\xi} \text{ent}_{\mu}^{\mathcal{F}, \mathbf{a}}(T, \xi)$$

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## REMARKS

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$$S_{\xi}^H(T, \{0, \dots, n-1\}, \epsilon, \delta) \approx e^{h_{\mu}(T) \cdot n}$$

- ▶ There are other popular scales such as the polynomial scale  $a_n(t) = n^t$ .

# A VARIATION OF SLOW ENTROPY

WHY IS THIS USEFUL?

- ▶ For a skew product  $R_\alpha \times_\tau T$  and a product partition  $\xi \times \eta$  of  $\mathbb{T} \times X$ , estimating  $S_{\xi \times \eta}^H(R_\alpha \times_\tau T, \{0, \dots, n-1\}, \epsilon, \delta)$  is difficult because it involves the statistics of

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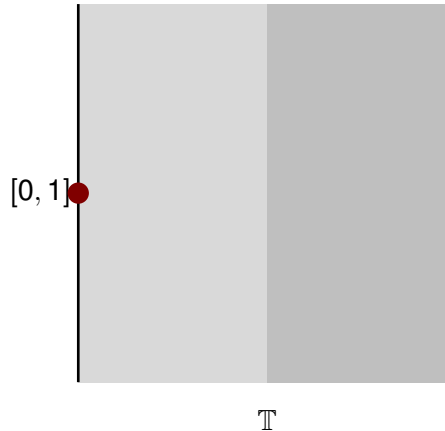
$$S_{\xi \times \eta}^H(R_\alpha \times_\tau T, \{0, q_n, 2q_n, \dots, (n-1)q_n\}, \epsilon, \delta) \approx S_\eta^H(T, \{0, \dots, n-1\}, \epsilon, \delta)$$

## A VARIATION OF SLOW ENTROPY

ITERATES OF  $R_\alpha \times_\tau T$  AND  $T$

The fundamental claim is that the iterates of  $(R_\alpha \times_\tau T)^{q_n}$  give us good information about the iterates of  $T$  ( $n$  iterates). Here  $|\alpha - p_n/q_n| < 1/(n^2 q_n)$ .

Visualization of  $(R_\alpha \times_\tau T)^{q_n} \curvearrowright [0, 1]^2$  with  $T(x) = \sqrt{x}$  on  $[0, 1]$  ( $\alpha = 1/\sqrt{26}$ )

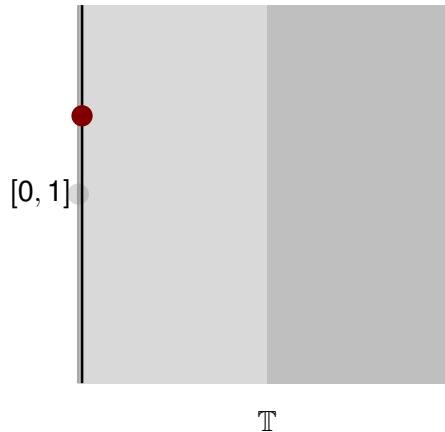


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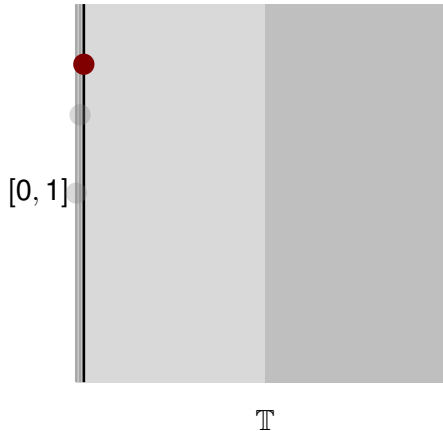


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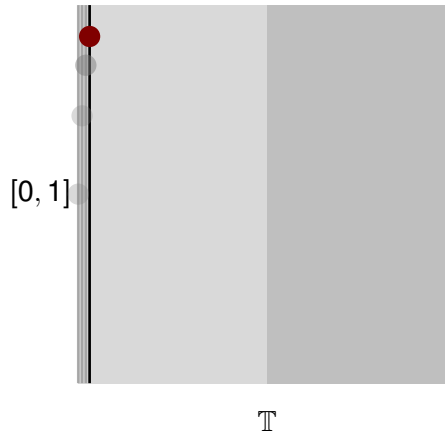


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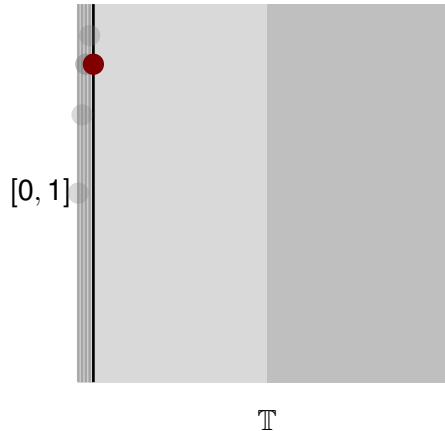


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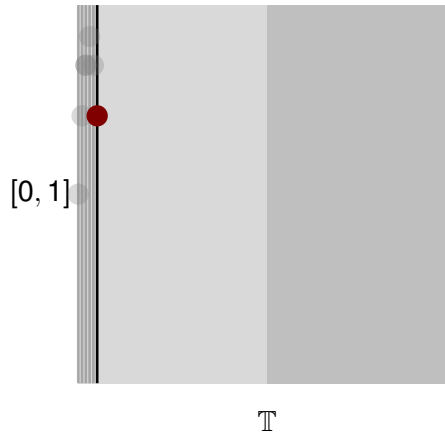


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## A VARIATION OF SLOW ENTROPY

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### Recall

Recall that  $q_n$  is defined by the condition that they are all odd numbers and  $|\alpha - p_n/q_n| < 1/(n^2 q_n)$ .

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Let  $T \curvearrowright (X, \mathcal{B}, \mu)$  be an automorphism. Then for every scale  $\mathbf{a}$  we have

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RETURN TO STEP 1

## Theorem (local version)

Let  $T \curvearrowright (X, \mathcal{B}, \mu)$  be an automorphism and let  $\xi$  be a finite partition of  $\mathbb{T} \times X$ . Then

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Let  $T \curvearrowright (X, \mathcal{B}, \mu)$  be a weakly mixing automorphism. Then there exists  $\mathbf{a}$  so that for every nontrivial partition  $\xi$  we have

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### Corollary

Let  $T \curvearrowright (X, \mathcal{B}, \mu)$  be a weakly mixing automorphism and choose  $\mathbf{a}$  as in the previous result. Let  $P \subset \mathbb{T} \times X$ . Then

$$\text{ent}_{m \times \mu}^{\mathcal{F}, \mathbf{a}}(R_\alpha \times_{\tau} T, \{P, \mathbb{T} \times X \setminus P\}) = 0$$

if and only if

# A VARIATION OF SLOW ENTROPY

## RETURN TO STEP 1

### Theorem (local version)

Let  $T \curvearrowright (X, \mathcal{B}, \mu)$  be an automorphism and let  $\xi$  be a finite partition of  $\mathbb{T} \times X$ . Then

$$\text{ent}_{m \times \mu}^{\mathcal{F}, \mathbf{a}}(R_\alpha \times_{\tau} T, \xi) = \text{esssup}_{\theta \in \mathbb{T}} \text{ent}_{\mu}^{\mathbf{a}}(T, \xi_{\theta})$$

### Lemma

Let  $T \curvearrowright (X, \mathcal{B}, \mu)$  be a weakly mixing automorphism. Then there exists  $\mathbf{a}$  so that for every nontrivial partition  $\xi$  we have

$$\text{ent}_{\mu}^{\mathbf{a}}(T, \xi) > 0$$

Essentially follows from existing results of Ferenczi and Lott.

### Corollary

Let  $T \curvearrowright (X, \mathcal{B}, \mu)$  be a weakly mixing automorphism and choose  $\mathbf{a}$  as in the previous result. Let  $P \subset \mathbb{T} \times X$ . Then

$$\text{ent}_{m \times \mu}^{\mathcal{F}, \mathbf{a}}(R_\alpha \times_{\tau} T, \{P, \mathbb{T} \times X \setminus P\}) = 0$$

if and only if  $P = B \times X$  for some  $B \in \mathcal{B}$  (up to a null set).

THE END

Thanks!<sup>1</sup>

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<sup>1</sup>Beamer theme: elegant slides. Tikz pictures: Kimi AI and Gemini AI