Medvedev degrees and subshifts

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Introduction

Theorem (Miller 2012)

On $\mathbb{Z}$, there are effective subshifts with all Π_1^0 Medvedev degrees.

Theorem (N.C. 2023)

The same holds replacing $\mathbb{Z}$ by a finitely generated infinite group with decidable word problem.

This talk will be based on:

- Nicanor Carrasco-Vargas. The geometric subgroup membership problem, March 2023. arXiv:2303.14820 [math],
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- Work in progress

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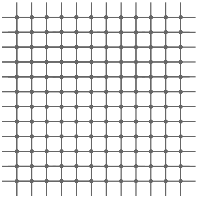
Outline

(boring) definitions for graphs



Definition

In a graph, degree of a vertex = number of incident edges
Locally finite graph = every vertex has finite degree

Informal definition

*A graph Γ is **highly computable** if it is locally finite and given some vertex v and radius $r \geq 0$, we can compute the finite subgraph induced by the associated ball.*

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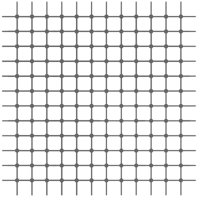

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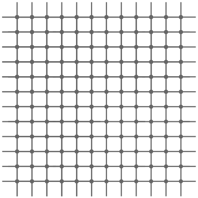

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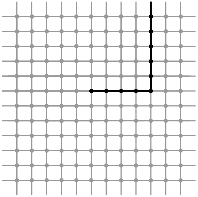
König's theorem

An **infinite path** on a graph Γ is a sequence of different vertices $v_0, v_1, \ldots,$ with $d(v_i, v_{i+1}) = 1$ for all i.



Theorem (König)

A locally finite and connected graph Γ has an infinite path $x : \mathbb{N} \to V(\Gamma)$ if and only if its vertex set is infinite.

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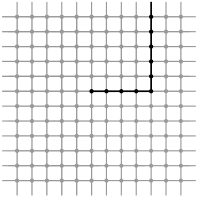

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Konig's theorem is computable?

Question

Is there an algorithm to compute infinite paths on graphs?

Proposition (Folklore?)

There is a highly computable graph with infinite paths, but all of them are uncomputable.

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What about Hamiltonian paths?

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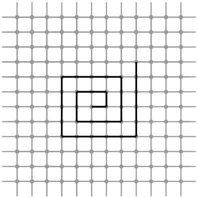
Hamiltonian infinite path = visits every vertex exactly once.



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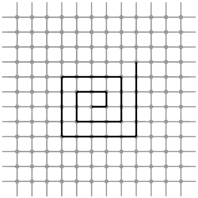

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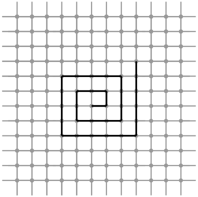

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Ends: a necessary condition

Remark

If a graph Γ admits an infinite Hamiltonian path, then it has one end.

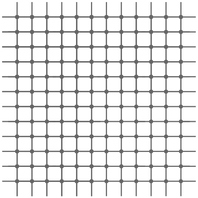
Question

What are these ends?

Ends

Definition

The number of ends of a locally finite graph Γ is the maximal number of infinite connected components of Γ that remain after removing finitely many vertices.



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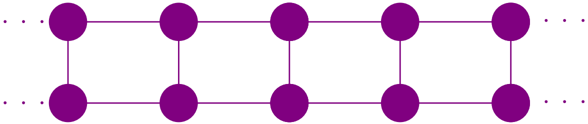

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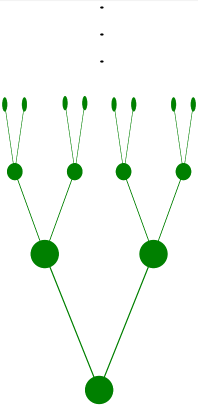

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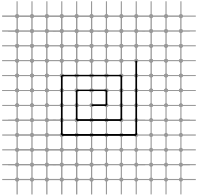

Let's relax paths to n-paths?

Definition

An infinite n-path is a sequence indexed by $\mathbb{N}$

$$v_0, v_1, v_2, v_3, \ldots$$

of different vertices where $d(v_i, v_{i+1})$ is at most n.

Theorem (N.C. 2023)

For $n \geq 3$, a locally finite graph Γ admits an infinite Hamiltonian n-path if and only if

1. *Γ is connected, and has countably many vertices,*
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A similar statement holds for bi-infinite paths.

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Theorem (N.C. 2023)

For $n \geq 3$, A highly computable graph admits an infinite Hamiltonian n-path if and only if it admits a computable one.
In this case, we can decide if a finite n-path can be extended to an infinite Hamiltonian n-path.

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The same hold for bi-infinite paths.

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What about Eulerian trails?

Definition

An infinite trail on a graph Γ is a sequence indexed by $\mathbb{N}$

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of alternating and incident vertices and edges. It is called **Eulerian** when it visits every *edge* exactly once.

Theorem (Pál Erdős, T. Grünwald, and E. Weiszfeld 1936)

A locally finite graph Γ admits an infinite Eulerian trail if and only if:

1. *Γ is connected, it has countably many edges.*
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Remark

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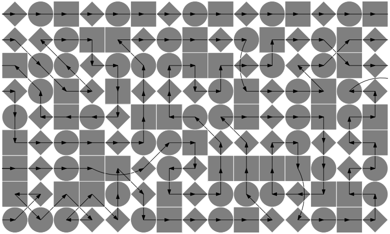
How?

1. Erase a finite trail or finite 3-path
2. Study the remaining connected components to find the right conditions.

Why?

Dynamical invariants of algorithmic nature



Subshfits on $\mathbb{Z}$

Definition

Take a finite alphabet A. Endow $A^{\mathbb{Z}}$ with the prodiscrete topology and the homeomorphism

$$\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$$
$$(x_i)_{i\in\mathbb{Z}} \mapsto (x_{i+1})_{i\in\mathbb{Z}}$$

This is called **full shift** on $\mathbb{Z}$.
Subshift=subsystem of the full shift.

Example: $A = \{\Diamond, \bigcirc, \Box\}$, one element in $A^{\mathbb{Z}}$ is

$$\dots \Diamond\bigcirc\Box\Diamond\bigcirc\Box\Diamond\bigcirc\Box\Diamond\bigcirc\Box\Diamond\bigcirc\Box\Diamond\bigcirc\Box \dots$$

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Take a finite alphabet A. Endow $A^{\mathbb{Z}}$ with the prodiscrete topology and the group action

$$\mathbb{Z} \curvearrowright A^{\mathbb{Z}}$$

$$(n \cdot x)_m = x_{m-n}$$

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Subshifts on $\mathbb{Z}^d$

Definition

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Subshifts on G

Definition

Take a finite alphabet A. Endow A^G with the prodiscrete topology and the shift action

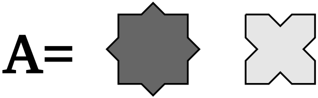
$$G \curvearrowright A^G$$

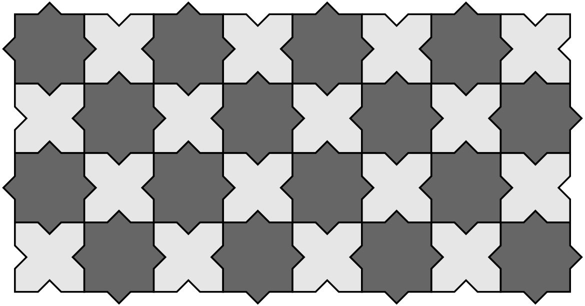
$$(g \cdot x)_h = x_{hg^{-1}}$$

This is called **full shift** on G.
Subshift=subsystem of the full shift.

Example: an infinite puzzle

Take A to be a finite set of puzzle pieces, and take X to be the set of *solutions* $x : \mathbb{Z}^2 \to A$ which respect the matching rules. Then X is a subshift on $\mathbb{Z}^2$.





A hard puzzle

Theorem (Hanf, Myers)

There is a puzzle (a finite set of square puzzle pieces A) such that every "solution" $x : \mathbb{Z}^2 \to A$ is uncomputable.

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Medvedev degrees, a complexity measure

Definition

Given $P, Q \subset A^{\mathbb{N}}$, we write

$$P \leq_{\mathfrak{M}} Q$$

if there is some computable functional Φ from Q to P (i.e. whose domain contains Q, and with $\Phi(Q) \subset P$).
We define $\equiv_{\mathfrak{M}}$ by $\leq_{\mathfrak{M}}$ and $\geq_{\mathfrak{M}}$
Medvedev degrees = equivalence classes by $\equiv_{\mathfrak{M}}$

Remark

Meaningful for mass problems, i.e. P = set of solutions of some problem.

Nice properties

1. The empty set is maximal for $\leq_{\mathfrak{M}}$
2. If P has a computable element, it is minimal for $\leq_{\mathfrak{M}}$. Its degree is denoted $0_{\mathfrak{M}}$.
3. $(\mathfrak{M}, \leq_{\mathfrak{M}})$ is a lattice!
4. $P \vee Q \equiv_{\mathfrak{M}} P \times Q$
5. $P \wedge Q \equiv_{\mathfrak{M}} P \sqcup Q$

Medvedev degrees vs entropy

Topological entorpy (for suitable G)

1. Measures uncertainty / information / more interpretations
2. Nonincreasing by factor maps
3. Conjugacy invariant
4. Takes values in $[0, \infty)$
5. Sends $\sqcup$ and $\times$ to max and sum

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The classification problem

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What are the possible entropies of this class of subshifts?

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Some classes of subshifts

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A subshift $X \subset A^G$ is called **of finite type** (SFT) if it can be defined by a finite set of local rules. Up to isomorphism, this is the same as a puzzle.

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The state of the art

- On $\mathbb{Z}$ all SFT's have computable points, and thus are $\equiv_{\mathfrak{M}} 0_{\mathfrak{M}}$
- On $\mathbb{Z}^d$, $d \geq 2$, there are nonempty SFT's with no computable point, that is, $>_{\mathfrak{M}} 0_{\mathfrak{M}}$ (Myers and Hanf, 1974)
- More generally, they attain all Π_1^0 degrees (Simpson 2014).

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For the known cases we observe a "0-1 law"

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Proof idea: Seward's theorem

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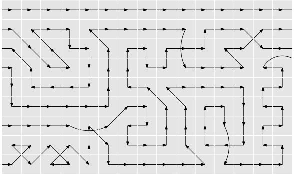
Some groups of interest do not contain $\mathbb{Z}$ as subgrouup.

Theorem (Seward 2014)

All finitely generated infinite groups admit a translation-like action by $\mathbb{Z}$.

This means a free action $\mathbb{Z} \curvearrowright G$ with a uniform bound on $d(g, g * 1)$ for $g \in G$.



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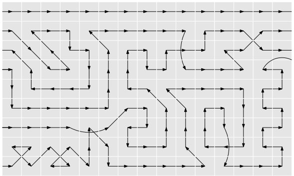

Computable Seward's theorem

Theorem

With the additional hypothesis of decidable word problem, the translation-like action by $\mathbb{Z}$ can be taken computable, and with decidable orbit membership problem.

Decidable orbit membership problem means that given two elements in G, we can decide if they lie in the same orbit under the action.

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For one ended groups, we can obtain the translation-like action from a computable bi-infinite and Hamiltonian 3-path. This was is the hard case, and for other groups it can be constructed algebraically. □

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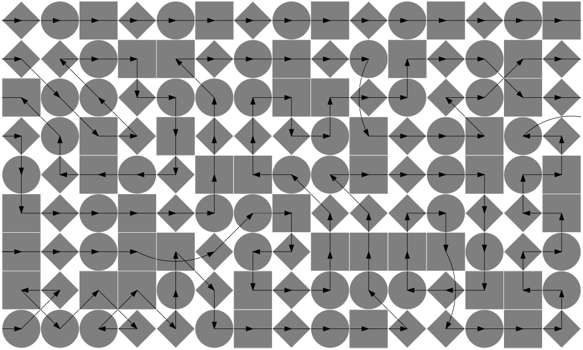
Proof Sketch

Claim

Given a subshift $X \subset A^{\mathbb{Z}}$, we can construct a subshift $X \subset B^{G}$ such that we preserve the Medvedev degree and the property of being effective. This is done describing translation-like actions on G.



Future work

Question

For effective subshifts, can we relax decidable word problem to recursively enumerable?

Problem

What happens with SFT's?

Thanks

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